CALCULUL ELEMENTAR AL ARIILOR SI AL VOLUMELOR

Calculați:

1) $\mathcal{A}(\Gamma_f)$, $x \in [0;1]$, $f(x) = \frac{x}{x+1}$

2) $\mathcal{A}(\Gamma_f)$, $x \in [-1;5]$, $f(x) = x^2 - 4x + 3$

3) $\mathcal{A}(\Gamma_{f,g})$, $x \in [0;4]$, $f(x) = -\sqrt{x}$, $g(x) = \sqrt{x}$

4) $\mathcal{A}(\Gamma_{f,g})$, $x \in [0;1]$, $f(x) = x^3$, $g(x) = x^2$

5) $\mathcal{V}(C_f)$, $x \in [0;1]$, $f(x) = x\sqrt{1-x}$

6) $\mathcal{V}(C_f)$, $x \in [1;2]$, $f(x) = e^{-x}$

7) $\mathcal{A}(\Gamma_{f,g})$, $x \in [-2;1]$, $f(x) = x^2+1$, $g(x) = 3-x$

8) $\mathcal{A}(\Gamma_f)$, $x \in [-1;1]$, $f(x) = \sqrt{4-x^2}$

9) $\mathcal{V}(C_f)$, $x \in [1;3]$, $f(x) = \frac{x}{\sqrt{x+1}}$

10) $\mathcal{A}(\Gamma_f)$, $x \in [0;9]$, $f(x) = \frac{x}{(x^2+1)(x^2+4)}$

11) $\mathcal{A}(\Gamma_{f,g})$, $x \in [0;4]$, $f(x) = x^2$, $g(x) = \sqrt{x}$

12) $\mathcal{A}(\Gamma_{f,g})$, $x \in [1;3]$, $f(x) = \frac{1}{x^2}$, $g(x) = x$

13) $\mathcal{V}(C_f)$, $x \in [1;e]$, $f(x) = \sqrt{\frac{\ln x}{x}}$

14) $\mathcal{V}(C_f)$, $x \in [0;1]$, $f(x) = x\sqrt{2-x^2}$

15) $\mathcal{V}(C_f)$, $x \in [0;1]$, $f(x) = \frac{x}{\sqrt{x^2+1}}$

16) $\mathcal{V}(C_f)$, $x \in [0;1]$, $f(x) = 1 - \sqrt{x}$

17) $\mathcal{V}(C_f)$, $x \in [0;1]$, $f(x) = \frac{x}{x+2}$

Calculați aria gt $f:[-1;5]\to\mathbb{R}$,

$f(x)=x^2-4x+3$

$E_1)\ A=\int_{-1}^{5}|x^2-4x+3|\,dx$

$E_2)\ x^2-4x+3=0 \Rightarrow x_{1,2}=\frac{4\pm\sqrt{16-12}}{2\cdot 1}=\frac{4\pm 2}{2}\begin{matrix}3\\1\end{matrix}$

$E_3)$

x	$[-1$		1		3		$5]$
x^2-4x+3	$+$	$+$	0	$-$	0	$+$	$+$

$A=\int_{-1}^{1}(x^2-4x+3)\,dx+\int_{1}^{3}(-x^2-4x-3)\,dx+\int_{3}^{5}(x^2-4x+3)\,dx$

$=\left(\frac{x^3}{3}\Big|_{-1}^{1}-4\cdot\frac{x^2}{2}\Big|_{-1}^{1}+3x\Big|_{-1}^{1}\right)+\dots\dots\dots$

Fie $f:[0;1]\to\mathbb{R}$, $f(x)=\frac{x}{x+1}$

Calculați aria dintre graficul lui f, axa Ox și dreptele $x=0$ și $x=1$

E1) $f(x)=\frac{x}{x+1}$

E2) $A=\int_a^b |f(x)|dx=\int_0^1 \left|\frac{x}{x+1}\right| dx$

E3) semnul pentru $\frac{x}{x+1}$ când $x\in[0;1]$

	-1		0		$1]$
x	$-$	$-$	0	$+$	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$\frac{x}{x+1}$	$+$	$\nexists$	$-$	0	$+$

$x=0\Rightarrow$ soluție

$x+1=0\Rightarrow x=-1$ soluție

$\forall x\in[0;1]\to\frac{x}{x+1}\geq 0\to\left|\frac{x}{x+1}\right|=\frac{x}{x+1}$

E4) $A=\int_0^1 \frac{x}{x+1}dx=\int_0^1 \frac{x+1-1}{x+1}dx=\int_0^1 1\,dx-\int_0^1\frac{dx}{x+1}$

$=x\Big|_0^1-\ln|x+1|\Big|_0^1=(1-0)-(\ln 2-\ln 1)=1-\ln 2$

Fie $f(x)=\frac{x}{(x^2+1)(x^2+4)}$, calculați aria dintre Gf, dreptele $x=0$ și $x=1$

E1) $A=\int_0^1 |f(x)|\,dx=\int_0^1 \left|\frac{x}{(x^2+1)(x^2+4)}\right| dx$

E2) Dar, $x\in[0;1] \Rightarrow \frac{x}{(x^2+1)(x^2+4)}\geq 0 \Rightarrow$

$\Rightarrow A=\int_0^1 \frac{x}{(x^2+1)(x^2+4)}\,dx$

E3) $\frac{x}{(x^2+1)(x^2+4)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}=$

$=\frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)}=\frac{Ax^3+4Ax+Bx^2+4B+}{}$

$=\frac{Ax^3+4Ax+Bx^2+4B+Cx^3+Cx+Dx^2+D}{(x^2+1)(x^2+4)}$

$=\frac{x^3(A+C)+x^2(B+D)+x(4A+C)+4B+D}{(x^2+1)(x^2+4)}$

E4) Egalăm coef de la numărat $\Rightarrow$ $\begin{cases} A+C=0 \Rightarrow C=-A \\ B+D=0 \Rightarrow D=-B \\ 4A+C=1 \Rightarrow 3A=1 \Rightarrow A=\frac{1}{3} \\ 4B+D=0 \Rightarrow 3B=0 \Rightarrow B=0 \end{cases}$

E5) $I=\int_0^1 \frac{\frac{1}{3}x+0}{x^2+1}dx+\int_0^1 \frac{-\frac{1}{3}x+0}{x^2+4}dx=$

$=\frac{1}{2}\int_0^1 \frac{x}{x^2+1}dx-\frac{1}{3}\int_0^1 \frac{x}{x^2+4}dx=\frac{1}{3}\cdot\frac{1}{2}\ln|x^2+1|\Big|_0^1-\frac{1}{3}\cdot\frac{1}{2}\ln|x^2+4$

$1\ln 2-\ln 5+\ln 4)$

Calculați volumul corpului obținut prin rotirea în jurul lui Ox a G_g, $g:[1,e]\to\mathbb{R}$,

$$g(x)=\sqrt{\frac{\ln x}{x}}$$

$$E_1)\ V=\pi\int_a^b g^2(x)dx=\pi\int_1^e \frac{\ln x}{x}dx$$

$$E_2)\ I=\int_1^e \frac{\ln x}{x}dx$$

$$f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}$$

$$g'(x)=\frac{1}{x}\Rightarrow g(x)=\ln x$$

$$I=\ln x\cdot\ln x\Big|_1^e-\int_1^e \frac{\ln x}{x}dx$$

$$I=1\cdot 1-0\cdot 0-I$$

$$\Rightarrow 2I=1\Rightarrow I=\frac{1}{2}$$

$$E_3)\ V=\pi\cdot\frac{1}{2}=\frac{\pi}{2}$$

Fie $f(x) = x\sqrt{2-x^2}$, $f:[0,1] \to \mathbb{R}$

a) Se cere volumul corpului obținut prin rotația în jurul lui Ox a G_f

$E_1)\ V = \pi \int_a^b f^2(x)\,dx$

$E_2)\ V = \pi \int_0^1 \left(x\cdot\sqrt{2-x^2}\right)^2 dx$

$V = \pi \int_0^1 x^2(2-x^2)\,dx$

$V = \pi \int_0^1 (2x^2 - x^4)\,dx = \pi\left(2\cdot\frac{x^3}{3}\Big|_0^1 - \frac{x^5}{5}\Big|_0^1\right)$

$V = \pi\left(\frac{2}{3}(1-0) - \frac{1}{5}(1-0)\right) = \pi\left(\frac{2}{3}-\frac{1}{5}\right) = \frac{7\pi}{15}$

Fie $f:[0,1]\to\mathbb{R}$, $f(x)=\frac{x}{\sqrt{x^2+1}}$. Găsiți volumul corpului obținut prin rotirea graficului lui f în jurul lui Ox

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$E_1)\ V=\pi\int_a^b f^2(x)\,dx$

$E_2)\ V=\pi\int_0^1\left(\frac{x}{\sqrt{x^2+1}}\right)^2dx=\pi\int_0^1\frac{x^2}{x^2+1}\,dx$

$E_3)\ I=\int_0^1\frac{x^2}{x^2+1}\,dx=\int_0^1\frac{x^2+1-1}{x^2+1}\,dx=$

$=\int_0^1\frac{x^2+1}{x^2+1}\,dx-\int_0^1\frac{1}{x^2+1}\,dx$

$=x\Big|_0^1-\frac{1}{1}\operatorname{arctg}\frac{x}{1}\Big|_0^1=$

$=1-0-(\operatorname{arctg}1-\operatorname{arctg}0)$

$=1-\left(\frac{\pi}{4}-0\right)=1-\frac{\pi}{4}=\frac{4-\pi}{4}$

$E_4)\ V=\pi\cdot\frac{4-\pi}{4}$

Fie $f:[0;1]\to\mathbb{R}$, $f(x)=1-\sqrt{x}$, calculați volumul obținut prin rotirea G_f

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$E_1)\ V=\pi\int_0^1 f^2(x)dx$

$E_2)\ V=\pi\int_0^1(1-\sqrt{x})^2dx=\pi\int_0^1(1-2\sqrt{x}+x)dx$

$=\pi\left(\int_0^1 1dx-2\int_0^1\sqrt{x}dx+\int_0^1 x dx\right)$

$E_3)\ V=\pi\left(x\Big|_0^1-2\cdot\int_0^1 x^{\frac{1}{2}}dx+\frac{x^2}{2}\Big|_0^1\right)$

$V=\pi\left((1-0)-2\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_0^1+\left(\frac{1}{2}-\frac{0}{2}\right)\right)$

$V=\pi\left(1-2\cdot x^{\frac{3}{2}}\cdot\frac{2}{3}\Big|_0^1+\frac{1}{2}\right)$

$V=\pi\left(1-\frac{4}{3}\left(1^{\frac{3}{2}}-0^{\frac{3}{2}}\right)+\frac{1}{2}\right)$

$V=\pi\left(\frac{6}{1}-\frac{2}{\frac{4}{3}}+\frac{3}{\frac{1}{2}}\right)=\pi\cdot\frac{6-8+3}{6}=\frac{\pi}{6}$

$f_1(x) = \frac{x}{x+2}$, calculați volumul corpului obținut prin rotirea graficului lui f_1 în jurul lui Ox,

$f_1 : [0;1] \to \mathbb{R}$

E1) $V = \pi \cdot \int_a^b f^2(x)\,dx$

E2) $V = \pi \int_0^1 \left(\frac{x}{x+2}\right)^2 dx = \pi \int_0^1 \frac{x^2}{x^2+4x+4}\,dx$

E3) integrală din fracție cu același grad la numărător și numitor, construim la numărător numitorul

E4) $V = \pi \int_0^1 \frac{x^2+4x+4-4x-4}{x^2+4x+4}\,dx = \pi\left(\int_0^1 \frac{x^2+4x+4}{x^2+4x+4}\,dx - 4\int_0^1 \frac{x+1}{x^2+4x+4}\,dx\right)$

$= \pi\left(\int_0^1 1\,dx - 4\underbrace{\int_0^1 \frac{x+1}{x^2+4x+4}\,dx}_{I_1}\right)$

E5) integrală din fracție cu gr numărăt. 1, gr numitorului 2, construim la numărător derivata de la numitor și folosim

$\int \frac{f'}{f} = \ln|f| + C \Rightarrow I_1 = \frac{1}{2}\int_0^1 \frac{2x+4-4+2}{x^2+4x+4}\,dx =$

$= \frac{1}{2}\left(\int_0^1 \frac{(x^2+4x+4)'}{x^2+4x+4} - 2\int_0^1 \frac{1}{x^2+4x+4}\,dx\right) = \frac{1}{2}\left(\ln|x^2+4x+4|\Big|_0^1 - 2\int_0^1 \frac{1}{(x+2)^2}\,dx\right) = \frac{1}{2}\left(\ln 9 - \ln 4 - 2\int_0^1 (x+2)^{-2}\,dx\right)$

$= \frac{1}{2}\ln\frac{9}{4} - \frac{1}{2}\cdot 2\cdot \frac{(x+2)^{-2+1}}{-2+1}\Big|_0^1 = \frac{1}{2}\ln\frac{9}{4} - \frac{(x+2)^{-1}}{-1}\Big|_0^1$

$= \frac{1}{2}\ln\frac{9}{4} - \left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{2}\ln\frac{9}{4} - \frac{1}{6}$

$V = \pi\left(1 - 2\ln\frac{9}{4} + \frac{2}{3}\right)$