# CALCULAREA LIMITELOR DE ȘIRURI CU AJUTORUL INTEGRALELOR

Calculați $\lim_{n\to\infty} x_n$ în situațiile următoare:

1) $x_n = \frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-2^2}} + \dots + \frac{1}{\sqrt{4n^2-n^2}}$

2) $x_n = \frac{1^2}{4n^3+1^3} + \frac{2^2}{4n^3+2^3} + \dots + \frac{n^2}{4n^3+n^3}$

3) $x_n = \sum_{k=1}^{n} \frac{n}{(n+k)\sqrt{n^2+(n+k)^2}}$

4) $x_n = \frac{1}{n^2}\left(e^{\frac{1^2}{n^2}} + 2e^{\frac{2^2}{n^2}} + \dots + ne^{\frac{n^2}{n^2}}\right)$

5) $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$

6) $x_n = \frac{1}{n\sqrt{n}} \sum_{k=1}^{n} \sqrt{k}$

7) $x_n = \frac{1^2}{4n^3+1^3} + \frac{2^2}{4n^3+2^3} + \dots + \frac{n^2}{4n^3+n^3}$

8) $x_n = n \cdot \sum_{k=1}^{n} \frac{1}{(n+k)^2}$

9) $x_n = \frac{1}{n^2} \sum_{k=1}^{n} k \cdot e^{\frac{k}{n}}$

10) $x_n = \frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \dots + \sqrt{1+\frac{n}{n}}\right)$

11) $x_n = \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2-k^2}$

12) $x_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2}$

13) $x_n = \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots + \frac{n}{n^2+n^2}$

Calculați $\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\dots+\frac{1}{\sqrt{4n^2-n^2}}\right)$

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E1) $x_n=\sum_{k=1}^{n}\frac{1}{\sqrt{4n^2-k^2}}$

E2) termenul general al sumei trebuie să depindă doar de $\frac{k}{n}$, adică nu are voie să fie $k$ izolat sau $n$ izolat și trebuie factor $\frac{1}{n}$ în fața sumei

E3) $x_n=\sum_{k=1}^{n}\frac{1}{n\sqrt{4-\left(\frac{k}{n}\right)^2}}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}}$

E4) $f:[0;1]\to\mathbb{R}$, $f(x)=\frac{1}{\sqrt{4-x^2}}$ (întotdeauna $f:[0;1]\to\mathbb{R}$ și în loc de $\frac{k}{n}$ punem $x$)

E5) $f$ continuă $\Rightarrow f$ integrabilă

E6) Fie $\Delta=\left(0<\frac{1}{n}<\frac{2}{n}<\dots<\frac{n}{n}\right)$ o diviziune a $[0;1]$ cu $\|\Delta\|=\frac{1}{n}\to 0$ și $\xi_k=\frac{k}{n}$ sistem de puncte intermediare (întotdeauna $E_6$ se scrie identic)

E7) $\Rightarrow \lim_{n\to\infty} x_n=\int_0^1 f(x)\,dx=\int_0^1\frac{1}{\sqrt{4-x^2}}\,dx=$

$=\arcsin\frac{x}{2}\Big|_0^1=\arcsin\frac{1}{2}-\arcsin 0=$

$=\frac{\pi}{6}-0=\frac{\pi}{6}$

Calculați $\lim_n x_n = ?$ unde

$$x_n = \frac{1^2}{4n^3+1^3} + \frac{2^2}{4n^3+2^3} + \dots + \frac{n^2}{4n^3+n^3}$$

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$E_1)\ x_n = \sum_{k=1}^{n} \frac{k^2}{4n^3+k^3}$

$E_2)\ x_n = \sum_{k=1}^{n} \frac{\left(\frac{k}{n}\right)^2 \cdot n^2}{n^3\left(4+\left(\frac{k}{n}\right)^3\right)} = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{\left(\frac{k}{n}\right)^2}{4+\left(\frac{k}{n}\right)^3}$

$$= \frac{1}{n} \sum_{k=1}^{n} \frac{\left(\frac{k}{n}\right)^2}{4+\left(\frac{k}{n}\right)^3}$$

$E_3)\ f: [0;1] \to \mathbb{R},\ f(x) = \frac{x^2}{4+x^3}$

$E_4)$ fie $\Delta = \left(0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n}{n}\right)$ o diviziune pt $[0;1]$ cu $\|\Delta\| = \frac{1}{n} \to 0$ și $\xi_k = \frac{k}{n}$ sistem de puncte intermediare

$E_5)\ \lim_{n\to\infty} x_n = \int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{4+x^3}\,dx =$

$$= \frac{1}{3}\int_0^1 \frac{3x^2}{4+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{(4+x^3)'}{4+x^3}\,dx =$$

$$= \frac{1}{3}\ln|4+x^3|\Big|_0^1 = \frac{1}{3}(\ln 5 - \ln 4) = \frac{1}{3}\ln\frac{5}{4}$$

Calculați: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(n+k)\sqrt{n^2+(n+k)^2}}$

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$E_1$) trebuie factor comun pe $\frac{1}{n}$ și termenul general depinde doar de $\frac{k}{n}$

$E_2$) $\frac{n}{(n+k)\sqrt{n^2+(n+k)^2}} = \frac{n}{n(1+\frac{k}{n})n\sqrt{1+(1+\frac{k}{n})^2}}$

$= \frac{1}{n(1+\frac{k}{n})\sqrt{1+(1+\frac{k}{n})^2}}$

$E_3$) Fie $f:[0;1]\to\mathbb{R}$, $f(x)=\frac{1}{(1+x)\sqrt{1+(1+x)^2}}$

$\Delta = (0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n}{n})$ cu $\|\Delta\| = \frac{1}{n} \to 0$

$\xi_k = \frac{k}{n}$ șirul de noduri intermediare

$E_4$) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+\frac{k}{n})\sqrt{1+(1+\frac{k}{n})^2}} = \int_0^1 f(x)\,dx$

$= \int_0^1 \frac{1}{(1+x)\sqrt{1+(1+x)^2}} \overset{\text{not } x+1=t}{\underset{dx=dt}{}} \int_1^2 \frac{1}{t\sqrt{1+t^2}}$

$= \int_1^2 \frac{1}{t^2\sqrt{1+\frac{1}{t^2}}}\,dt = \int_1^2 \frac{\frac{1}{t^2}}{\sqrt{1+(\frac{1}{t})^2}}\,dt$

$1 \quad 1/t \quad -\frac{1}{t^2}dt = du \Rightarrow I = \int_{\frac{1}{2}}^{1} \frac{-du}{\sqrt{1+u^2}}$

Calculați:

$$\lim_{n\to\infty} \frac{1}{n^2}\left(e^{\frac{1^2}{n^2}} + 2e^{\frac{2^2}{n^2}} + \dots + ne^{\frac{n^2}{n^2}}\right)$$

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$E_1)$ Fie $x_n = \frac{1}{n^2}\left(e^{\frac{1^2}{n^2}} + 2e^{\frac{2^2}{n^2}} + \dots + ne^{\frac{n^2}{n^2}}\right) \Rightarrow$

$$\Rightarrow x_n = \sum_{k=1}^{n} \frac{1}{n^2} k e^{\frac{k^2}{n^2}} = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k}{n} \cdot e^{\frac{k^2}{n^2}} \Rightarrow$$

$$\Rightarrow x_n = \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \cdot e^{\left(\frac{k}{n}\right)^2}$$

$E_2)$ Fie $f:[0;1]\to\mathbb{R}$, $f(x) = xe^{x^2}$ cont. pe $[0;1]$

$\Rightarrow f$ integrabilă pe $[0;1]$ și

$\Delta = \left(0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n}{n}\right)$ o diviziune

a lui $[0;1]$ cu $\|\Delta\| = \frac{1}{n} \to 0$ și

$\xi_k = \frac{k}{n}$ sistem de puncte intermediare

$E_3) \Rightarrow \lim_n x_n = \int_0^1 f(x)\,dx = \int_0^1 x e^{x^2} dx$

$E_4)$ not. $t = x^2 / (\,)' \Rightarrow dt = 2x\,dx \Rightarrow x\,dx = \frac{dt}{2}$

$x = 0 \Rightarrow t = 0$, $x = 1 \Rightarrow t = 1$

$$\Rightarrow I = \int_0^1 e^t \cdot \frac{dt}{2} = \frac{1}{2} e^t \Big|_0^1 = \frac{1}{2}(e-1) = \frac{e-1}{2}$$

Calculați $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+n}\right)$

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E1) $x_n=\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+n}$

$x_n=\sum_{k=1}^{n}\frac{1}{n+k}=\sum_{k=1}^{n}\frac{1}{n\left(1+\frac{k}{n}\right)}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}$

E2) Fie $f:[0;1]\to\mathbb{R}$, $f(x)=\frac{1}{1+x}$

$f$ continuă pe $[0;1]\Rightarrow f$ integrabilă pe $[0;1]$

E3) Fie $\Delta=\left(0<\frac{1}{n}<\frac{2}{n}<\dots<\frac{n}{n}\right)$ o diviziune pt $[0;1]$ cu $\|\Delta\|=\frac{1}{n}\to 0$ și $\xi_k$ sist de pct intermediare, $\xi_k=\frac{k}{n}$

E4) $\lim_{n\to\infty}x_n=\int_0^1 f(x)\,dx=\int_0^1\frac{1}{x+1}\,dx=$

$=\ln(x+1)\Big|_0^1=\ln 2-\ln 1=\ln 2$