PRIMITIVELE FUNCȚIILOR RAȚIONALE SIMPLE (FĂRĂ ÎMPĂRȚIRE)

Calculați:

1) $\int \frac{1}{3x+1}\,dx$

2) $\int \frac{1}{-5x+7}\,dx$

3) $\int \frac{1}{9-x^2}\,dx$

4) $\int \frac{1}{x^2+4x+3}\,dx$

5) $\int \frac{5}{2x^2+7x+3}\,dx$

6) $\int \frac{3x}{4x^2+2x+5}\,dx$

7) $\int \frac{7x+4}{5x^2+8x+3}\,dx$

8) $\int \frac{x^2+3x+8}{x^2+5x+1}\,dx$

9) $\int \frac{5x^2+7x+3}{5x^2+x-7}\,dx$

10) $\int \frac{-2x^2+4x-5}{3x^2+4x+5}\,dx$

11) $\int \frac{x+3}{x^2+4x+3}\,dx$

12) $\int \frac{x}{x^2+5}\,dx$

13) $\int \frac{x^2}{x^2+1}\,dx$

14) $\int \frac{x}{x^2+4x+3}\,dx$

Prof. Ovidiu Bădescu

Calculați $\int \frac{1}{x^2+4x+3}\,dx$

E_1) Deoarece $gr\,P=0$, $gr\,Q=2$ se formează forma canonică: $ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a}$

E_2) în cazul nostru $\Rightarrow \Delta=16-4\cdot 3=4$

$$x^2+4x+3=1\left(x+\frac{4}{2\cdot 1}\right)^2-\frac{4}{4\cdot 1}$$

$$=(x+2)^2-1$$

E_3) $I=\int \frac{1}{(x+2)^2-1}\,dx$

E_4) notăm $t=x+2 \mid (\,)' \Rightarrow dt=dx$

$$\Rightarrow I=\int \frac{1}{t^2-1}\,dt=\frac{1}{2\cdot 1}\ln\left|\frac{t-1}{t+1}\right|+C$$

E_5) $I=\frac{1}{2}\ln\left|\frac{x+2-1}{x+2+1}\right|+C$

$$I=\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|+C$$

Prof. Ovidiu Bădescu

Calculați: $\int \frac{3x}{4x^2+2x+5}\,dx$

E_1) dacă numărătorul este de gr I, numitorul este de gr II facem la numărător derivata numitorului

E_2) $(4x^2+2x+5)' = 8x+2$

E_3) $I = 3\int \frac{x}{4x^2+2x+5}\,dx = \frac{3}{8}\int \frac{8x}{4x^2+2x+5}\,dx =$

$= \frac{3}{8}\int \frac{8x+2-2}{4x^2+2x+5}\,dx = \frac{3}{8}\left(\underbrace{\int \frac{8x+2}{4x^2+2x+5}\,dx}_{I_1} - 2\underbrace{\int \frac{dx}{4x^2+2x+5}}_{I_2}\right)$

E_4) pt $I_1 = \int \frac{8x+2}{4x^2+2x+5}\,dx = \int \frac{(4x^2+2x+5)'}{4x^2+2x+5}\,dx =$

$= \ln|4x^2+2x+5| + C$ pt că am folosit

$\int \frac{f'}{f} = \ln|f| + C$

E_5) pt $I_2 = \int \frac{1}{4x^2+2x+5}\,dx \overset{\text{formă canonică}}{=\!=\!=} \int \frac{dx}{4\left(x+\frac{2}{2\cdot 4}\right)^2 - \frac{-76}{4\cdot 4}} =$

$= \int \frac{dx}{4\left(x+\frac{1}{4}\right)^2 + \frac{19}{4}} = \frac{1}{4}\int \frac{1}{\left(x+\frac{1}{4}\right)^2 + \frac{19}{16}}\,dx$

$t = x+\frac{1}{4} \,|\,()' \Rightarrow dt = dx \Rightarrow I_2 = \frac{1}{4}\int \frac{1}{t^2+\frac{19}{16}}\,dt =$

$= \frac{1}{4}\cdot\frac{1}{\sqrt{\frac{19}{16}}}\cdot \operatorname{arctg}\frac{t}{\sqrt{\frac{19}{16}}} + C \Rightarrow I_2 = \frac{1}{\sqrt{19}}\operatorname{arctg}\frac{4\left(x+\frac{1}{4}\right)}{\sqrt{19}} + C$

E_6) $I = \frac{3}{8}\left(\ln|4x^2+2x+5| - 2\cdot\frac{1}{\sqrt{19}}\operatorname{arctg}\frac{4x+1}{\sqrt{19}}\right) + C$

Prof. Ovidiu Bădescu

Calculați: $\int \frac{7x+4}{5x^2+8x+3}dx$

$E_1) (5x^2+8x+3)' = 10x+8$

$E_2) I = 7\int \frac{x+\frac{4}{7}}{5x^2+8x+3}dx = \frac{7}{10}\int \frac{10x+\frac{40}{7}}{5x^2+8x+3}dx$

$= \frac{7}{10}\int \frac{10x+8+\frac{40}{7}-8}{5x^2+8x+3}dx = \frac{7}{10}\left(\underbrace{\int \frac{10x+8}{5x^2+8x+3}dx}_{I_1} - \frac{16}{7}\underbrace{\int \frac{dx}{5x^2+8x+3}}_{I_2}\right)$

$E_3) I_1 = \int \frac{(5x^2+8x+3)'}{5x^2+8x+3}dx = \ln|5x^2+8x+3| + C$

$E_4) I_2 = \int \frac{1}{5x^2+8x+3}dx$, numărătorul → număr, numitorul → grad II

⇒ formă canonică $\Rightarrow a(x+\frac{b}{2a})^2 - \frac{\Delta}{4a}$

$\Rightarrow I_2 = \int \frac{1}{5(x+\frac{8}{10})^2 - \frac{64-60}{20}}dx = \int \frac{1}{5(x+\frac{4}{5})^2 - \frac{1}{5}}dx$

notăm $t = x+\frac{4}{5} \mid ()' \Rightarrow dt = dx$

$I_2(t) = \int \frac{1}{5t^2-\frac{1}{5}}dt = \frac{1}{5}\int \frac{1}{t^2-\frac{1}{25}}dt$

$= \frac{1}{5}\cdot\frac{1}{2\cdot\frac{1}{5}}\ln\left|\frac{t-\frac{1}{5}}{t+\frac{1}{5}}\right| + C = \frac{1}{2}\ln\left|\frac{5t-1}{5t+1}\right| + C$

$I_2(x) = \frac{1}{2}\ln\left|\frac{5x+4-1}{5x+4+1}\right| + C = \frac{1}{2}\ln\left|\frac{5x+3}{5x+5}\right| + C$

$E_5) I = \frac{7}{10}\left(\ln|5x^2+8x+3| - \frac{16}{7}\cdot\frac{1}{2}\ln\left|\frac{5x+3}{5x+5}\right|\right) + C$

Prof. Ovidiu Bagescu

Calculați: $\int \frac{5x^2+7x+3}{5x^2+x-7}\,dx$

E_1) dacă avem grade egale, încerc să formez la numărător exact numitorul

E_2) $I = \int \frac{5x^2+x-7+6x+10}{5x^2+x-7}\,dx =$

$= \int \frac{5x^2+x-7}{5x^2+x-7}\,dx + \int \frac{6x+10}{5x^2+x-7}\,dx = x + \underbrace{\int \frac{6x+10}{5x^2+x-7}\,dx}_{I_1}$

E_3) I_1 are grad numitor 2, iar numărător 1

$\Rightarrow (5x^2+x-7)' = 10x+1$

$I_1 = 6\int \frac{x+\frac{10}{6}\cdot\frac{5}{3}}{5x^2+x-7}\,dx = \frac{6}{10}\int \frac{10x+\frac{100}{6}}{5x^2+x-7}\,dx$

$= \frac{3}{5}\int \frac{10x+1-\frac{3}{1}+\frac{50}{3}}{5x^2+x-7}\,dx = \frac{3}{5}\left(\underbrace{\int \frac{10x+1}{5x^2+x-7}\,dx}_{I_2} + \frac{47}{3}\underbrace{\int \frac{dx}{5x^2+x-7}}_{I_3} \right)$

E_4) $I_2 = \int \frac{10x+1}{5x^2+x-7}\,dx = \int \frac{(5x^2+x-7)'}{5x^2+x-7} =$

$= \ln|5x^2+x-7| + C$

E_5) $I_3 = \int \frac{dx}{5x^2+x-7} = \int \frac{dx}{5(x+\frac{1}{10})^2 - \frac{141}{20}} \xrightarrow[dx=dt]{x+\frac{1}{10}=t}$

$I_3(x) = \frac{1}{5}\int \frac{dt}{t^2-\frac{141}{100}} = \frac{1}{5}\cdot\frac{1}{2\sqrt{\frac{141}{100}}}\cdot \ln\frac{t-\frac{\sqrt{141}}{10}}{t+\frac{\sqrt{141}}{10}} + C$

$= \frac{1}{5}\cdot\frac{10}{2\sqrt{141}} \ln\frac{10(x+\frac{1}{10})-\sqrt{141}}{10(x+\frac{1}{10})+\sqrt{141}} + C = \frac{\sqrt{141}}{141}\ln\frac{10x+1-\sqrt{141}}{10x+1+\sqrt{141}} + C$

E_6) $I = x + \frac{3}{5}\ln|5x^2+x-7| + \frac{3}{5}\cdot\frac{47}{3}\cdot\frac{\sqrt{141}}{141}\ln\frac{10x+1-\sqrt{141}}{10x+1+\sqrt{141}} + C$

Prof. Ovidiu Badescu

Calculați $\int \frac{-2x^2+4x-5}{3x^2+4x+5}dx$

$E_1) I = -\frac{1}{3}\int \frac{6x^2-12x+15}{3x^2+4x+5}dx =$

$= -\frac{1}{3}\int \frac{2(3x^2+4x+5)-20x-5}{3x^2+4x+5}dx =$

$= -\frac{1}{3}\left(\int \frac{2(3x^2+4x+5)}{3x^2+4x+5}dx - \int \frac{20x+5}{3x^2+4x+5}dx\right)$

$= -\frac{1}{3}\left(2x - 20\underbrace{\int \frac{x+\frac{1}{4}}{3x^2+4x+5}dx}_{I_1}\right)$

$E_2)$ not $I_1 = \int \frac{x+\frac{1}{4}}{3x^2+4x+5}$ $\overset{(3x^2+4x+5)'=6x+4}{=\!=\!=}$ $\frac{1}{6}\int \frac{6x+\frac{3}{2}}{3x^2+4x+5}dx$

$= \frac{1}{6}\int \frac{6x+4+\frac{3}{2}-4}{3x^2+4x+5}dx = \frac{1}{6}\left(\underbrace{\int \frac{6x+4}{3x^2+4x+5}dx}_{I_2} - \frac{5}{2}\underbrace{\int \frac{1}{3x^2+4x+5}dx}_{I_3}\right)$

$E_3)$ not $I_2 = \int \frac{6x+4}{3x^2+4x+5}dx = \int \frac{(3x^2+4x+5)'}{3x^2+4x+5}dx = \ln|3x^2+4x+5| + C$

$E_4)$ not $I_3 = \int \frac{1}{3x^2+4x+5}dx = \int \frac{1}{3(x+\frac{4}{6})^2-\frac{44}{12}}dx = \int \frac{dx}{3(x+\frac{2}{3})^2-\frac{11}{3}}$

$E_5)$ not $x+\frac{2}{3} = t \mid (\,)' \Rightarrow dx = dt$

$I_3(x) = \frac{1}{3}\int \frac{dt}{t^2-\frac{11}{9}} = \frac{1}{3}\cdot\frac{1}{2\sqrt{\frac{11}{9}}}\ln\frac{t-\frac{\sqrt{11}}{3}}{t+\frac{\sqrt{11}}{3}} + C = \frac{1}{3}\cdot\frac{3}{2\sqrt{11}}\ln\frac{3t-\sqrt{11}}{3t+\sqrt{11}} + C$

$\Rightarrow I_3(x) = \frac{1}{2\sqrt{11}}\ln\frac{3(x+\frac{2}{3})-\sqrt{11}}{3(x+\frac{2}{3})+\sqrt{11}} + C = \frac{\sqrt{11}}{22}\ln\frac{3x+2-\sqrt{11}}{3x+2+\sqrt{11}} + C$

$E_6) I(x) = -\frac{1}{3}\left(2x - 20\left(\frac{1}{6}\left(\ln|3x^2+4x+5| - \frac{5}{2}\cdot\frac{\sqrt{11}}{22}\ln\frac{3x+2\sqrt{11}}{3x+2+\sqrt{11}}\right)\right)\right) + C$

Prof. Ovidiu Pădeanu

Calculați: $\int \frac{x+3}{x^2+4x+3}\,dx$

E_1) Deoarece $gr\,P=1$, $gr\,Q=2$, formăm la numărător derivata numitorului

E_2) $(x^2+4x+3)' = 2x+4$

E_3) $I = \frac{1}{2}\int \frac{2x+6}{x^2+4x+3}\,dx = \frac{1}{2}\left(\int \frac{2x+4}{x^2+4x+3}\,dx + \int \frac{2\,dx}{x^2+4x+3}\right)$

E_4) Folosim $\int \frac{\varphi'}{\varphi} = \ln|\varphi| + C$

$$\Rightarrow \int \frac{2x+4}{x^2+4x+3}\,dx = \int \frac{(x^2+4x+3)'}{x^2+4x+3}\,dx = \ln|x^2+4x+3| + C$$

E_5) $\int \frac{1}{x^2+4x+3}\,dx \xrightarrow[a(x+\frac{b}{2a})^2 - \frac{\Delta}{4a}]{\text{formă canonică}} \int \frac{1}{1\cdot(x+\frac{4}{2\cdot 1})^2 - \frac{4}{4}}\,dx$

$$= \int \frac{1}{(x+2)^2 - 1}\,dx \xrightarrow[dx=dt]{x+2=t} \int \frac{1}{t^2-1}\,dt$$

$$= \frac{1}{2\cdot 1}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{x+2-1}{x+2+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + C$$

$$= \frac{1}{2}\left(\ln|x^2+4x+3| + 2\cdot\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|\right) + C$$

Prof. Ovidiu Bădescu

III 2.4/232 Calculați: $\int \frac{x}{x^2+5}\,dx$

E1) numărătorul gr I și numitorul gr II, formăm la numărător derivata de la numitor

E2) $I = \frac{1}{2}\int \frac{2x}{x^2+5}\,dx = \frac{1}{2}\int \frac{(x^2+5)'}{x^2+5}\,dx$

$= \frac{1}{2}\ln|x^2+5| + C$

Dar, $x^2+5 \geq 0 \Rightarrow I = \frac{1}{2}\ln(x^2+5)+C$

Obs.: s-a folosit $\int \frac{\varphi'}{\varphi} = \ln|\varphi| + C$

Prof. Ovidiu Băgescu

Calculați: $\int \frac{x^2}{x^2+1}\,dx$

E_1) deoarece $\text{gr } P = \text{gr } Q$, încercăm să formăm la numărător numitorul

E_2) $I = \int \frac{x^2+1-1}{x^2+1}\,dx = \int \frac{x^2+1}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx =$

$= \int 1\,dx - \int \frac{1}{x^2+1}\,dx$

E_3) $I = x - \frac{1}{1}\operatorname{arctg}\frac{x}{1} + C$

$I = x - \operatorname{arctg} x + C$

Obs.: s-a folosit $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$

Prof. Ovidiu Badescu

Calculați: $\int \frac{x}{x^2+4x+3}dx$

E_1) Deoarece gr $P=1$, gr $Q=2 \Rightarrow$ formăm la numărător derivata numitorului

Calculăm derivata numitorului

$$(x^2+4x+3)' = 2x+4$$

E_2) $I = \int \frac{x}{x^2+4x+3}dx = \frac{1}{2}\cdot\int \frac{2x}{x^2+4x+3}dx =$

$$= \frac{1}{2}\int \frac{2x+4-4}{x^2+4x+3}dx = \frac{1}{2}\left(\underbrace{\int \frac{2x+4}{x^2+4x+3}dx}_{I_1} - 4\underbrace{\int \frac{dx}{x^2+4x+3}}_{I_2}\right)$$

E_3) Fie $I_1 = \int \frac{2x+4}{x^2+4x+3}dx = \int \frac{(x^2+4x+3)'}{x^2+4x+3}dx =$

$= \ln|x^2+4x+3| + C$ fie că $\int \frac{f'}{f} = \ln|f| + C$

E_4) Fie $I_2 = \int \frac{dx}{x^2+4x+3}$ folosim forma canonică

$ax^2+bx+c = a(x+\frac{b}{2a})^2 - \frac{\Delta}{4a} \Rightarrow I_2 = \int \frac{dx}{1(x+2)^2 - \frac{4}{4}}$

$= \int \frac{dx}{(x+2)^2-1}$, not $x+2 = t \mid (\,)' \Rightarrow dx = dt$

$\Rightarrow I_2 = \int \frac{dt}{t^2-1} = \frac{1}{2\cdot 1}\ln\left|\frac{t-1}{t+1}\right| + C \Rightarrow$

$\Rightarrow I_2 = \frac{1}{2}\ln\left|\frac{x+2-1}{x+2+1}\right| + C = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + C$

E_5) $I = I_1 + I_2 = \ln|x^2+4x+3| - \frac{4}{2}\ln\left|\frac{x+1}{x+3}\right| + C$

Prof. Ovidiu Bădescu