DEMONSTRAREA INEGALITĂȚILOR FOLOSIND MONOTONIA

Arătați că:

1) $\frac{x}{1+x^2} < \operatorname{arctg} x, \forall x > 0$

2) $e^x \geq x+1, \forall x \in \mathbb{R}$

3) $\ln x \leq \frac{x}{e}, \forall x > 0$

4) $\ln(x+\sqrt{1+x^2}) < x, \forall x > 0$

5) $(x+1)^{2012} - 2012x - 1 \geq 0, \forall x \in \mathbb{R}$

6) $\ln(1+x) \leq x, \forall x \geq 0$

7) $x \operatorname{arctg} x - \ln(1+x^2) \geq 0, \forall x \in \mathbb{R}$

8) $e^x \geq ex, \forall x > 0$

9) $\frac{x}{1+x} \leq \ln(1+x), \forall x > -1$

10) $\frac{x}{1+x^2} - \frac{x^2}{2} \leq \operatorname{arctg} x, \forall x \in \mathbb{R}_+$

11) $\cos x \geq 1 - \frac{x^2}{2}, \forall x \in \mathbb{R}$

12) $x + \frac{x^3}{6} \leq \arcsin x, \forall x \in [0;1]$

13) $\ln(1+x^2) < x \operatorname{arctg} x, \forall x > 0$

14) $\operatorname{arctg} x < x, \forall x > 0$

15) $\sin x \leq x, \forall x \geq 0$

Prof. Ovidiu Bădescu

Arătați că: $\frac{x}{1+x^2} < \operatorname{arctg} x, \forall x > 0$

E_1) $\frac{x}{1+x^2} - \operatorname{arctg} x < 0, \forall x > 0$

E_2) $f: (0;\infty) \to \mathbb{R}$, $f(x) = \frac{x}{1+x^2} - \operatorname{arctg} x$

E_3) $f'(x) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} - \frac{1}{1+x^2}$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} - \frac{\frac{1+x^2}{1}}{1+x^2} = \frac{-2x^2}{(1+x^2)^2}$$

$f'(x) = 0 \Rightarrow -2x^2 = 0 \Rightarrow x = 0$

E_4)

x	$(0$
f'	$-$
f	$\searrow$

$\lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \left(\frac{x}{1+x^2} - \operatorname{arctg} x\right) \overset{0-0}{=\!=} 0$

E_5) din tabel $\Rightarrow f \searrow$ și $\lim\limits_{\substack{x\to 0\\ x>0}} f(x) = 0 \Rightarrow$

$\Rightarrow f(x) < 0, \forall x > 0$

$\Rightarrow \frac{x}{\operatorname{arctg} x} - \operatorname{arctg} x < 0 \Rightarrow \frac{x}{1+x^2} < \operatorname{arctg} x, \forall x > 0$

Prof. Ovidiu Bădescu

Arătați că $e^x \geq x+1, \forall x \in \mathbb{R}$

E_1) inegalitate folosind monotonia

E_2) $e^x - x - 1 \geq 0, \forall x \in \mathbb{R}$

E_3) $f(x) = e^x - x - 1$, $f'(x) = e^x - 1$

E_4) $f'(x) = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$

E_5)

x	$-\infty$		-1		0		1		$+\infty$
f'		$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
f		$\searrow$		$\searrow$	0	$\nearrow$		$\nearrow$	

$f'(-1) = e^{-1} - 1 = \frac{1}{e} - 1 < 0 \Rightarrow f'(x) < 0, \forall x \in (-\infty; 0)$

$\Rightarrow f \searrow$ pe $x \in (-\infty; 0)$

$f'(1) = e - 1 > 0 \Rightarrow f'(x) > 0, \forall x \in (0; \infty)$

$\Rightarrow f \nearrow$ pe $x \in (0; \infty)$

E_6) $f(0) = e^0 - 0 - 1 = 0$

E_7) Din tabel $\Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$

$\Rightarrow e^x - x - 1 \geq 0, \forall x \in \mathbb{R}$

$e^x \geq x + 1$

Prof. Ovidiu Bădescu

Arătați că $\ln x \leq \frac{x}{e}$, $\forall x > 0$

E_1) $\ln x \leq \frac{x}{e} \Leftrightarrow \ln x - \frac{x}{e} \leq 0$, $\forall x > 0$

E_2) Fie $g : (0, \infty) \to \mathbb{R}$, $g(x) = \ln x - \frac{x}{e}$

E_3) $g'(x) = \frac{1}{x} - \frac{1}{e} \cdot x' = \frac{1}{x} - \frac{1}{e}$

$g'(x) = 0 \Rightarrow \frac{1}{x} = \frac{1}{e} \Rightarrow x = e$

E_4)

x	0		e		∞
g'	///	$+$	0	$-$	
g	///	$\nearrow$	0	$\searrow$	

$g'(1) = \frac{1}{1} - \frac{1}{e} = 1 - \frac{1}{e} > 0 \Rightarrow g'(x) > 0 \Rightarrow$

$g \nearrow$, $\forall x \in (0, e)$

$g'(3) = \frac{1}{3} - \frac{1}{e} = \frac{e-3}{3e} < 0 \Rightarrow g'(x) < 0, \forall x \in (e, \infty)$

$\Rightarrow g \searrow$, $\forall x \in (e, \infty)$

$g(e) = \ln e - \frac{e}{e} = 1 - 1 = 0$

E_5) Din tabel $\Rightarrow g(x) \leq 0 \Rightarrow$

$\ln x - \frac{x}{e} \leq 0 \Rightarrow \ln x \leq \frac{x}{e}$

Prof. Ovidiu Badescu

Arătați că: $\ln(x+\sqrt{1+x^2}) < x,\ \forall x>0$

$E_1)\ \ln(x+\sqrt{1+x^2}) - x < 0,\ \forall x > 0$

$f:(0,\infty)\to\mathbb{R},\ f(x)=\ln(x+\sqrt{1+x^2})-x$

$E_2)\ f'(x)=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)-1$

$$f'(x)=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)-1$$

$$f'(x)=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}-1$$

$$f'(x)=\frac{1}{\sqrt{1+x^2}}-1=\frac{1-\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$E_3)\ f'(x)=0 \Rightarrow 1=\sqrt{1+x^2}\ /()^2 \Rightarrow 1=1+x^2$

$\Rightarrow x^2=0 \Rightarrow x=0 \notin D_f$

$E_4)$

x	0	∞
f'	$-$	$-$
f	$\searrow$	

Pt a găsi semnul lui $f'(x)$, dăm o valoare

$$f'(1)=\frac{1-\sqrt{2}}{\sqrt{2}}<0 \Rightarrow f'(x)<0,\ \forall x\in(0,\infty)$$

$\Rightarrow f\searrow$ pe $(0,\infty)$

$E_5)\ \lim\limits_{\substack{x\to 0\\ x>0}} f(x)=\lim\limits_{\substack{x\to 0\\ x>0}}\left(\ln(x+\sqrt{1+x^2})-x\right)\overset{\ln 1-0}{=\!=} 0,\ f\searrow$

$\Rightarrow f(x)<0 \Rightarrow \ln(x+\sqrt{1+x^2})-x<0 \Rightarrow \ln(x+\sqrt{1+x^2})<x$

Prof. Ovidiu Bădescu

Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=(x+1)^{2012}-2012x-1$ (5)

Arătați că $f(x)\geq 0,\ \forall x\in\mathbb{R}$

E_1) Ca teorie, folosim „demonstrarea unor inegalități cu monotonia"

E_2) $f'(x)=2012(x+1)^{2011}\cdot(x+1)'-2012$

$f'(x)=2012\left[(x+1)^{2011}-1\right]$

E_3) $f'(x)=0\Rightarrow(x+1)^{2011}=1\Rightarrow x+1=1\Rightarrow x=0$

E_4)

x	$-\infty$		0		$+\infty$
f'		$-$	0	$+$	
f	$\searrow$		0	$\nearrow$	

$f'(-1)=2012\left[(-1+1)^{2011}-1\right]=2012\left[0-1\right]<0$

$\Rightarrow f'(x)<0,\ \forall x\in(-\infty;0)\Rightarrow f_{\searrow}\to m(-\infty;0)$

Analog, $f'(1)=2012\left[2^{2011}-1\right]>0$

$\Rightarrow f'(x)>0,\ \forall x\in(0;\infty)\Rightarrow f_{\nearrow}\to m(0;\infty)$

E_5) $f(0)=1^{2012}-0-1=0$

E_6) Din tabel $\Rightarrow f'(x)\geq 0,\ \forall x\in\mathbb{R}$

Prof. Ovidiu Bădescu

Fie $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x \operatorname{arctg} x - \ln(1+x^2)$

Arătați că $f(x) \geq 0, \forall x \in \mathbb{R}$

E_1) se încadrează în tipul: „demonstrarea unor inegalități folosind monotonia"

E_2) $f'(x) = \operatorname{arctg} x + x \cdot \frac{1}{1+x^2} - \frac{1}{1+x^2} \cdot 2x$

$f'(x) = \operatorname{arctg} x + \frac{x}{1+x^2} - \frac{2x}{1+x^2} \Rightarrow f'(x) = \operatorname{arctg} x - \frac{x}{1+x^2}$

E_3) $f'(x) = 0$, nu not găsi rădăcinile și nici semnul, calculez $f''(x) = \frac{1}{1+x^2} - \frac{1(1+x^2) - x \cdot 2x}{(1+x^2)^2}$

$f''(x) = \frac{1+x^2 - 1 - x^2 + 2x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} \geq 0$ și

$f''(x) = 0 \Leftrightarrow x = 0$

E_4)

x		0	
f''	$+ + + + + + 0$		$+ + + + +$
f'	$- - - - - 0$		$+ + + + +$
f	↘	0	↗

$f''(x) \geq 0$ și $f''(x) = 0 \Leftrightarrow x = 0 \Rightarrow f'$ ↗

Calculez $f'(0) = \operatorname{arctg} 0 - 0 = 0$

f' ↗ pe $x \in (-\infty; 0)$, $f'(x) = 0 \Rightarrow f'(x) < 0$ pe $(-\infty; 0)$

f' ↗ pe $x \in (0; \infty)$, $f'(x) = 0 \Rightarrow f'(x) > 0$ pe $(0; \infty)$

$f'(x) < 0$ pe $(-\infty, 0) \Rightarrow f$ ↘ pe $(-\infty; 0)$

$f'(x) > 0$ pe $(0; \infty) \Rightarrow f$ ↗ pe $(0; \infty)$

$\Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$

Arătați că: $e^x \geq ex, \forall x > 0$

E1) $e^x - ex \geq 0, \forall x > 0$

E2) $f:(0;\infty) \to \mathbb{R}$, $f(x) = e^x - ex$

$f'(x) = e^x - e$; $f'(x) = 0 \Rightarrow e^x = e$

$\Rightarrow x = 1$

E3)

x	0		1	
f'	//	$---$	0	$+++$
f	//	$\searrow$	0	$\nearrow$

$f'\left(\frac{1}{2}\right) = e^{\frac{1}{2}} - e = \sqrt{e} - e < 0$

$f'(2) = e^2 - e > 0$

$f(1) = e^1 - e \cdot 1 \Rightarrow f(1) = 0$

E4) Din tabel $\Rightarrow f(x) \geq 0, \forall x \in (0;\infty)$

Prof. Ovidiu Bădescu

Arătați că: $\frac{x}{1+x} \leq \ln(1+x), \forall x > -1$

E_1) $\frac{x}{1+x} - \ln(1+x) \leq 0, \forall x > -1$

E_2) $f(x) = \frac{x}{1+x} - \ln(1+x), \quad f:(-1,\infty) \to \mathbb{R}$

E_3) $f'(x) = \frac{1\cdot(1+x) - x\cdot 1}{(1+x)^2} - \frac{1}{1+x} = \frac{1}{(1+x)^2} - \frac{1+x}{1+x}$

$f'(x) = \frac{1-1-x}{(1+x)^2} \Rightarrow f'(x) = \frac{-x}{(1+x)^2}$

$f'(x) = 0 \Leftrightarrow x = 0$

E_4)

x	-1		0	
f'	‖	$+$	0	$-$
f	‖	$\nearrow$	0	$\searrow$

$f(0) = 0 - \ln 1 = 0$

din tabel $\Rightarrow f(x) \leq 0, \forall x > -1$

$\Rightarrow \frac{x}{1+x} - \ln(1+x) \leq 0, \forall x > -1$

$\frac{x}{1+x} \leq \ln(1+x), \forall x > -1$

Prof. Ovidiu Badescu

Arătați că: $\frac{x}{1+x^2} - \frac{x^2}{2} \leq \operatorname{arctg} x, \forall x \geq 0$

$E_1)$ $\frac{x}{1+x^2} - \frac{x^2}{2} - \operatorname{arctg} x \leq 0, \forall x \in [0;\infty)$

$E_2)$ $f: [0;\infty) \to \mathbb{R}$, $f(x) = \frac{x}{1+x^2} - \frac{x^2}{2} - \operatorname{arctg} x$

$E_3)$ $f'(x) = \frac{1(1+x^2) - x \cdot 2x}{(1+x^2)^2} - \frac{1}{2} \cdot 2x - \frac{1}{1+x^2}$

$f'(x) = \frac{1-x^2}{(1+x^2)^2} - x - \frac{(1+x^2)}{1+x^2}$

$f'(x) = \frac{1-x^2-(1+x^2)}{(1+x^2)^2} - x = \frac{-2x^2}{(1+x^2)^2} - x$

$f'(x) = \frac{-2x^2 - x(1+x^2)^2}{(1+x^2)^2} \leq 0, \forall x \in [0;\infty)$

$\not\exists\ f'(x) = 0$ pt $x = 0$

$E_4)$

x	$[0$	
f'	0	$-\ -\ -\ -$
f		$\searrow$

$\Rightarrow f_S \searrow$

Cum $f(0) = 0 - 0 - \operatorname{arctg} 0 = 0$ $\Rightarrow$

$\Rightarrow f(x) \leq 0, \forall x \in [0;\infty)$

$\Rightarrow \frac{x}{1+x^2} - \frac{x^2}{2} - \operatorname{arctg} x \leq 0, \forall x \in [0;\infty)$

Prof. Ovidiu Bădescu

Arătați că: $\ln(1+x^2) < x \operatorname{arctg} x, \forall x > 0$

$E_1)$ $\ln(1+x^2) - x \cdot \operatorname{arctg} x < 0, \forall x > 0$

$E_2)$ $f: (0;\infty) \to \mathbb{R}$, $f(x) = \ln(1+x^2) - x \operatorname{arctg} x$

$f'(x) = \frac{1}{1+x^2} \cdot 2x - (1 \cdot \operatorname{arctg} x + x \cdot \frac{1}{1+x^2})$

$f'(x) = \frac{2x}{1+x^2} - \operatorname{arctg} x - \frac{x}{1+x^2}$

$f'(x) = \frac{x}{1+x^2} - \operatorname{arctg} x$

$E_3)$ $f''(x) = \frac{1 \cdot (1+x^2) - x \cdot 2x}{(1+x^2)^2} - \frac{1}{1+x^2}$

$f''(x) = \frac{1+x^2 - 2x^2 - 1 - x^2}{(1+x^2)^2} = \frac{-2x^2}{(1+x^2)^2}$

$E_4)$ $f''(x) = 0 \Rightarrow -2x^2 = 0 \Rightarrow x = 0 \notin A_f$

$\Rightarrow f''(x) \leq 0, \forall x \in (0;\infty) \Rightarrow f' \searrow$

$E_5)$

x	0	
f''		$-$
f'	0	$\searrow$
f		$\searrow$

$E_6)$ $\lim\limits_{\substack{x \to 0 \\ x > 0}} f'(x) = \lim\limits_{\substack{x \to 0 \\ x > 0}} \left(\frac{x}{1+x^2} - \operatorname{arctg} x \right) = 0$

Cum $f' \searrow \Rightarrow f'(x) < 0, \forall x \in (0;\infty)$

$\Rightarrow f \searrow$

$E_7)$ $\lim\limits_{x \to 0} f(x) = \lim\limits_{\substack{x \to 0 \\ x > 0}} (\ln(1+x^2) - x \operatorname{arctg} x) = 0$

$\Rightarrow f(x) < 0, \forall x \in (0;\infty)$

Prof. Ovidiu Bădescu

Arătați că:

$$\cos x \geq 1 - \frac{x^2}{2}, \forall x \in \mathbb{R}$$

$E_1)$ $\cos x - 1 + \frac{x^2}{2} \geq 0, \forall x \in \mathbb{R}$

$E_2)$ $f(x) = \cos x - 1 + \frac{x^2}{2}$, $f'(x) = -\sin x + x$

$f''(x) = -\cos x + 1$

$f''(x) = 0 \Rightarrow \cos x = 1 \Rightarrow$

$\Rightarrow x \in \{\arccos 1 + 2k\pi / k \in \mathbb{Z}\} \cup \{-\arccos 1 + 2k\pi / k \in \mathbb{Z}\}$

$\Rightarrow x \in \{0 + 2k\pi / k \in \mathbb{Z}\} \cup \{-0 + 2k\pi / k \in \mathbb{Z}\}$

$x \in \{0 + 2k\pi / k \in \mathbb{Z}\}$

$E_3)$

x	$\ldots$	-4π	-2π	0	2π	4π	$\ldots$
f''		$+0+0+0+0+0+$					
f'		$-\!-\!-/-/-0/+/+/+$					
f		$\searrow$		0		$\nearrow$	

$E_4)$ Dar, $\cos x \in [-1,1] \Rightarrow 1 - \cos x \geq 0 \Rightarrow f''(x) \geq 0$

$\Rightarrow f' \nearrow$

$E_5)$ Dar, $f'(0) = -\sin 0 + 0 = 0$ $\Bigg\} \Rightarrow$

$\Rightarrow f'(x) < 0$ pe $(-\infty, 0) \Rightarrow f \searrow$

$f'(x) > 0$ pe $(0, \infty) \Rightarrow f \nearrow$

$E_6)$ $f(0) = \cos 0 - 1 + 0 = 0 \Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$

Prof. Oriu Badescu

Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x\operatorname{arctg}x-\ln(1+x^2)$, arătați că $f(x)\geq 0$, $\forall x\in\mathbb{R}$

$E_1)$ $f'(x)=1\cdot\operatorname{arctg}x+x\cdot\frac{1}{1+x^2}-\frac{1}{1+x^2}\cdot 2x=\operatorname{arctg}x+\frac{x}{1+x^2}-\frac{2x}{1+x^2}$

$\Rightarrow f'(x)=\operatorname{arctg}x-\frac{x}{1+x^2}$, nu putem găsi rădăcinile lui $f'(x)$, nici semnul, calculăm $f''(x)$

$E_2)$ $f''(x)=\left(\operatorname{arctg}x-\frac{x}{1+x^2}\right)'=\frac{1+x^2}{1+x^2}-\frac{1\cdot(1+x^2)-x\cdot 2x}{(1+x^2)^2}=$

$=\frac{1+x^2-1-x^2+2x^2}{(1+x^2)^2}=\frac{2x^2}{(1+x^2)^2}$, $f''(x)\geq 0$ $\forall x$

$f'(x)=0$ doar dacă $x=0\Rightarrow f'_S\nearrow$

$E_3)$

x			0		
f''	$+$	$+$	0	$+$	$+$
f'	$-$	$-$	0	$+$	$+$
f	$\searrow$	$\searrow$	0	$\nearrow$	$\nearrow$

$f''(0)=0$, $f'(0)=\operatorname{arctg}0-0=0$, $f'_S\nearrow$ $\Big\}\xrightarrow{\text{tabel}}$

$f'(x)<0$ pe $(-\infty;0)\Rightarrow f_S\searrow$ pe $(-\infty;0)$

$f'(x)>0$ pe $(0;\infty)\Rightarrow f_S\nearrow$ pe $(0;\infty)$

$f(0)=0\operatorname{arctg}0-\ln 1=0$

$E_4)$ $f(0)=0$, $f_S\searrow$ pe $(-\infty;0)$, $f_S\nearrow$ pe $(0;\infty)$

$\xrightarrow{\text{tabel}}$ $f(x)\geq 0$, $\forall x\in\mathbb{R}$

Prof. Ovidiu Badescu

Fie $f:\mathbb{R}\setminus\{-1\}\to\mathbb{R}$, $f(x)=\frac{x^2}{x+1}$

Arătați că $f(x)\le -4$, $\forall x<-1$

$E_1)$ $f(x)\le -4 \Leftrightarrow \frac{x^2}{x+1}\le -4 \Leftrightarrow \frac{x^2}{x+1}+4\le 0$

$\Leftrightarrow \frac{x^2+4x+4}{x+1}\le 0$

$E_2)$ Fie $g:(-\infty,-1)\to\mathbb{R}$, $g(x)=\frac{x^2+4x+4}{x+1}$

$E_3)$ $g'(x)=\frac{(2x+4)(x+1)-(x^2+4x+4)\cdot 1}{(x+1)^2}=$

$=\frac{2x^2+2x+4x+4-x^2-4x-4}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}$

$E_4)$ $g'(x)=0\Rightarrow x^2+2x=0\Rightarrow x(x+2)=0\to\begin{cases}x=0\\x=-2\end{cases}$

$E_5)$

x		-2		-1		0
g'	$+$	0	$-$	$\nexists$	$+$	0
g	$\nearrow$	0	$\searrow$	$\nexists$	$\nearrow$	

Pt semnul lui g', calculăm $g'(-3)=\frac{9-6}{4}>0$

$\Rightarrow g'(x)>0$, $\forall x\in(-\infty,-2)$, analog $g'(-\frac{3}{2})<0$

$\Rightarrow g'(x)<0$, $\forall x\in(-2,-1)\Rightarrow g$ $\nearrow$ pe $(-\infty,-2)$

$g\searrow$ pe $(-2,-1)$

$E_6)$ $g(-2)=\frac{4-8+4}{-2+1}=0$, din tabelul de la $E_5)$

$\Rightarrow f(x)\le -4$, $\forall x\in(-\infty,-1)$