MULȚIMEA VALORILOR UNEI FUNCȚII. IMAGINEA LUI f

Găsiți mulțimea valorilor funcției $f: D_f \to \mathbb{R}$.

1) $f(x) = x^3 - 3x + 2$

2) $f(x) = \frac{2x+1}{x-3}$

3) $f(x) = x^3 - 3x + 4$, $f([-1;3]) = ?$

4) $f(x) = x^3 - 3x - 1$, $f([-1;5]) = ?$

5) $f(x) = \frac{x-1}{e^x}$

6) $f(x) = (x+2) \cdot e^{\frac{1}{x}}$

7) $f(x) = -4x + 3$, $f([0;4]) = ?$

8) $f(x) = \operatorname{arctg}(x+2) - \operatorname{arctg} x$

9) $f(x) = \frac{1}{x^2 - x + 1}$

10) $f(x) = \frac{2x}{x^2+1}$

11) $f(x) = -x^2$, $f([0;1]) = ?$

12) $f: [0;1] \to \mathbb{R}$, $f(x) = \frac{e^x}{x+2}$, x care: $\frac{3}{e} \leq \frac{1}{f(x)} \leq 2$

13) $a = ?$ dacă $\operatorname{Im} f = [-3; \infty)$, $f(x) = x^2 + ax + 1$

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Găsiți mulțimea valorilor lui $f(x)=x^3-3x+2$, $f:\mathbb{R}\to\mathbb{R}$

$E_1)\ f'(x)=3x^2-3$

$E_2)\ f'(x)=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

$E_3)$

x	$-\infty$		-1		1		∞
f'	$+$	$+$	0	$-$	0	$+$	$+$
f	$-\infty$	$\nearrow$	4	$\searrow$	0	$\nearrow$	∞

$E_4)\ \lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(x^3-3x+2)\overset{-\infty+\infty}{=\!=\!=}$

$=\lim\limits_{x\to-\infty} x^3\left(1-\frac{3}{x^2}+\frac{2}{x^3}\right)\overset{-\infty(1-0+0)}{=\!=\!=}-\infty$

$f(-1)=-1+3+2=4$

$f(1)=1-3+2=0$

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}(x^3-3x+2)\overset{\infty-\infty}{=\!=\!=}\lim\limits_{x\to\infty} x^3\left(1-\frac{3}{x^2}+\frac{2}{x^3}\right)$

$=\infty$

$E_5)\ \mathrm{Im} f=(-\infty;4]\cup[0;4]\cup[0;\infty)=\mathbb{R}$

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(2)

Fie $f: D_f \to \mathbb{R}$, $f(x) = \frac{2x+1}{x-3}$, să se afle $\operatorname{Im} f$

E_1) Cond $x-3 \neq 0 \Rightarrow x \neq 3 \Rightarrow D_f = (-\infty; 3) \cup (3; \infty)$

E_2)

x		3	
$f'(x)$	$-\ -\ -\ -\ -$	$\nexists$	$-\ -\ -\ -\ -$
$f(x)$	$2 \searrow -\infty$		$\infty \searrow 2$

$$f'(x) = \left(\frac{2x+1}{x-3}\right)' = \frac{2(x-3) - (2x+1)\cdot 1}{(x-3)^2} = \frac{2x-6-2x-1}{(x-3)^2} = \frac{-7}{(x-3)^2}$$

$\Rightarrow f'(x) < 0, \forall x \in D_f \Rightarrow f \searrow$

E_3) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \frac{2x+1}{x-3} = 2$

$\lim\limits_{\substack{x \to 3 \\ x < 3}} f(x) = \lim\limits_{\substack{x \to 3 \\ x < 3}} \frac{2x+1}{x-3} \overset{\frac{7}{0_-}}{=\!=} -\infty$

$\lim\limits_{\substack{x \to 3 \\ x > 3}} f(x) = \lim\limits_{\substack{x \to 3 \\ x > 3}} \frac{2x+1}{x-3} \overset{\frac{7}{0_+}}{=\!=} \infty$

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{2x+1}{x-3} = 2$

E_4) $\operatorname{Im} f = (-\infty; 2) \cup (2; \infty)$

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$f(x) = x^3 - 3x - 1$, să se afle imaginea lui $[-1;5]$

$E_1)$ $f'(x) = 3x^2 - 3$

$f'(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$E_2)$

x	$[-1$		1		$5]$
f'	$+ \quad +$	0	$- - - - - -$	0	$+ \quad + \quad +$
f	$\nearrow$	1	$\searrow$	-3	$\nearrow$ 109

$E_3)$ $f([-1;5]) =$ reuniunea tuturor intervalelor pe care f strict monotonă

$E_4)$ $f(-1) = (-1)^3 - 3 \cdot (-1) - 1 = -1 + 3 - 1 = 1$

$f(1) = 1 - 3 - 1 = -3$

$f(5) = 5^3 - 3 \cdot 5 - 1 = 125 - 15 - 1 = 109$

$E_5)$ $f([-1;5]) = [-3;1] \cup [-3;109] = [-3;109]$

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Găsiți imaginea funcției $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{x-1}{e^x}$

E_1) Vom face tabelul de variație al lui f căci $\operatorname{Im} f$ este reuniunea tuturor intervalelor lui f pe care f e strict monotonă

E_2) $f'(x)=\frac{1\cdot e^x-(x-1)e^x}{e^{2x}}=\frac{e^x(1-x+1)}{e^{2x}}=\frac{2-x}{e^x}$

E_3) $f'(x)=0\Rightarrow x=2$

E_4)

x	$-\infty$		2		∞
f'	$+$	$+$	0	$-$	$-$
f	$-\infty$	$\nearrow$	$\frac{1}{e^2}$	$\searrow$	0

$\lim\limits_{x\to-\infty}\frac{x-1}{e^x}\overset{\frac{-\infty}{0}}{=\!=}-\infty$

$f(2)=\frac{2-1}{e^2}=\frac{1}{e^2}$

$\lim\limits_{x\to\infty}\frac{x-1}{e^x}\overset{\frac{\infty}{\infty}}{=\!=}\overset{l'H}{\lim\limits_{x\to\infty}}\frac{1}{e^x}\overset{\frac{1}{\infty}}{=\!=}0$

E_5) $\operatorname{Im} f=\left(-\infty;\frac{1}{e^2}\right]\cup\left(0;\frac{1}{e^2}\right]=$

$=\left(-\infty;\frac{1}{e^2}\right]$

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Fie $f: \mathbb{R}^* \to \mathbb{R}$, $f(x) = (x+2) \cdot e^{\frac{1}{x}}$. Găsiți imaginea lui f

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E1) $f'(x) = (x+2)' e^{\frac{1}{x}} + (x+2)(e^{\frac{1}{x}})' =$

$= 1 \cdot e^{\frac{1}{x}} + (x+2) e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)' =$

$= e^{\frac{1}{x}} + (x+2) e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$

$= e^{\frac{1}{x}} \left(1 - \frac{x+2}{x^2}\right) = e^{\frac{1}{x}} \cdot \frac{x^2 - x - 2}{x^2}$

E2) $f'(x) = 0 \Rightarrow$ $e^{\frac{1}{x}} = 0$ fals

sau

$x^2 - x - 2 = 0$

$\Rightarrow x_{1,2} = \frac{1 \pm \sqrt{9}}{2} \begin{matrix} \nearrow 2 \\ \searrow -1 \end{matrix}$

E3)

x	$-\infty$		-1		0		2		∞
f'		$+$	0	$-$	$\nexists$	$-$	0	$+$	
f	$-\infty$	$\nearrow$	$\frac{1}{e}$	$\searrow$	0	$\nearrow$	$4\sqrt{e}$	$\nearrow$	∞

$f(-1) = 1 e^{-1} = \frac{1}{e}$

$f(2) = 4 e^{\frac{1}{2}} = 4\sqrt{e}$

E4) $\lim_{x \to -\infty} (x+2) e^{\frac{1}{x}} = -\infty$; $\lim_{\substack{x \to 0 \\ x < 0}} (x+2) e^{\frac{1}{x}} = 0$

$\lim_{x \to 0} (x+2) e^{\frac{1}{x}} = \infty$; $\lim_{x \to \infty} (x+2) e^{\frac{1}{x}} = \infty$

$\ldots = (-\infty, -1] \cup (0, \frac{1}{e}] \cup [4\sqrt{e}, \infty) = (-\infty, \frac{1}{e}] \cup [4\sqrt{e}, \infty)$

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$f: [0;4] \to \mathbb{R}$, $f(x) = -4x+3$

Se cere: mulțimea valorilor lui f

$E_1)$ $f'(x) = (-4x+3)' = -4 \cdot 1 + 0 = -4 < 0$

$\Rightarrow f \searrow$

$E_2)$

x	$[0$		$4]$
$f'(x)$		$-\ -\ -\ -$	
$f(x)$	$[3$	$\searrow$	$-13]$

$E_3)$ $f(0) = 3$; $f(4) = -4 \cdot 4 + 3 = -13$

$E_4)$ $\mathrm{Im} f = [-13; 3]$

I^* nu e voie $[3;-13]$ nr că, la t interval $[a;b]$ trebuie ca $a < b$ $*$

Prof. Ovidiu Bădoiu

Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\operatorname{arctg}(x+2)-\operatorname{arctg}x$

Se cere: $0<f(x)\le\frac{\pi}{2}$, $\forall x\in\mathbb{R}$

E_1) Ar trebui să arătăm că $\operatorname{Im}f=\left(0,\frac{\pi}{2}\right]$

E_2) $f'(x)=\frac{1}{1+(x+2)^2}\cdot(x+2)'-\frac{1}{1+x^2}=$

$=\frac{1+x^2-(1+(x+2)^2)}{(1+(x+2)^2)(1+x^2)}=$

$=\frac{1+x^2-1-x^2-4x-4}{(1+(x+2)^2)(1+x^2)}=\frac{-4x-4}{(1+(x+2)^2)(1+x^2)}$

E_3) $f'(x)=0\Leftrightarrow -4x-4=0\Leftrightarrow x=-1$

E_4)

x		-1	
f'	$+\;+\;+\;+$	0	$-\;-\;-\;-$
f	$0\nearrow$	$\frac{\pi}{2}$	$\searrow 0$

E_5) $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}(\operatorname{arctg}(x+2)-\operatorname{arctg}x)=$

$-\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=0$

Știm că: $\operatorname{arctg}(-\infty)=-\frac{\pi}{2}$

Analog, $\lim\limits_{x\to\infty}f(x)=\frac{\pi}{2}-\frac{\pi}{2}=0$; $f(-1)=$

$f(-1)=\operatorname{arctg}(+1)-\operatorname{arctg}(-1)=+\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

$f(-1)\to(0,\frac{\pi}{2}]$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x^2-x+1}$, se care $0 \le f(x) \le \frac{4}{3}$

E_1) trebuie găsită mulțimea valorilor lui f

E_2) $f'(x) = \frac{1' \cdot (x^2-x+1) - 1 \cdot (x^2-x+1)'}{(x^2-x+1)^2} = \frac{-2x+1}{(x^2-x+1)^2}$

E_3) $f'(x) = 0 \Rightarrow -2x+1 = 0 \Rightarrow -2x = -1 \Rightarrow x = \frac{1}{2}$

E_4)

x		$\frac{1}{2}$	
f'	$+ + + + + + 0$		$- - - - - -$
f	$\nearrow$	$\frac{1}{\infty}$	$\searrow$

E_5) $\lim\limits_{x \to -\infty} \frac{1}{x^2-x+1} = \frac{1}{\infty} = 0$

$f(\frac{1}{2}) = \frac{1}{\frac{1}{4} - \frac{1}{2} + 1} = \frac{1}{\frac{1-2+4}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

$\lim\limits_{x \to \infty} \frac{1}{x^2-x+1} = \frac{1}{\infty} = 0$

E_6)

x	$-\infty$		$\frac{1}{2}$		∞
f'		$+ + + + + + 0$		$- - - - - -$	
f	0	$\nearrow$	$\frac{4}{3}$	$\searrow$	0

$\text{Im } f =$ reuniunea tuturor intervalelor pe care f e

$\text{Im } f = (0, \frac{4}{3}] \cup (0, \frac{4}{3}] = (0, \frac{4}{3}]$

Prof. Ovidiu Badescu

Găsiți imaginea $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{2x}{x^2+1}$

$E_1)$ $f'(x)=\left(\frac{2x}{x^2+1}\right)'=\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}=\frac{-2x^2+2}{(x^2+1)^2}$

$E_2)$ $f'(x)=0 \Rightarrow -2x^2+2=0 \Rightarrow -2x^2=-2 \Rightarrow x^2=1$

$\Rightarrow x=\pm 1$

$E_3)$

x	$-\infty$		-1		1		∞
f'		$-$	0	$+$	0	$-$	
f	0	$\searrow$	-1	$\nearrow$	1	$\searrow$	0

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\frac{2x}{x^2+1}\overset{grP<grQ}{=\!=\!=}0$

$f(-1)=\frac{-2}{2}=-1$; $f(1)=\frac{2}{2}=1$; $\lim\limits_{x\to\infty} f(x)=0$

$E_4)$ $Im f=[-1;0)\cup[-1;1]\cup(0;1]=[-1;1]$

Prof. Ovidiu Badescu

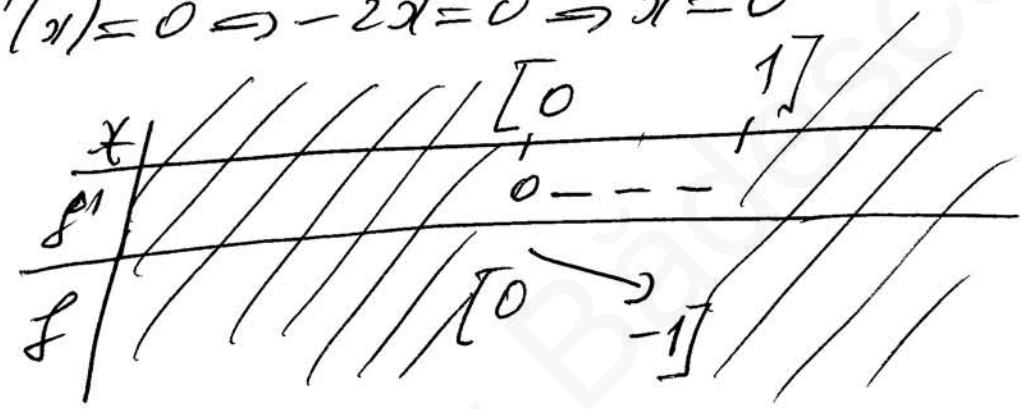
$f:[0;1]\to\mathbb{R}$, $f(x)=-x^2$. Se cere mulțimea valorilor funcției f

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$E_1)$ $f'(x)=(-x^2)'=-2x$

$E_2)$ $f'(x)=0 \Leftrightarrow -2x=0 \Rightarrow x=0$

$E_3)$



$E_4)$ $f(0)=-0^2=0$

$f(1)=-1^2=-1$

$E_5)$ mulțimea valorilor este imaginea lui f

= reuniunea tuturor intervalelor pe care f este strict monotonă $\Rightarrow [-1;0]$

Prof. Ovidiu Bădescu

Fie $f:[0;1]\to\mathbb{R}$, $f(x)=\frac{e^x}{x+2}$

Arătați că $\frac{3}{e}\le\frac{1}{f(x)}\le 2, \forall x\in[0;1]$

E_1) Relația $\frac{3}{e}\le\frac{1}{f(x)}\le 2 \iff$

$\Rightarrow \frac{e}{3}\ge f(x)\ge\frac{1}{2}$ adică

ar trebui arătat că $\frac{1}{2}\le f(x)\le\frac{e}{3}$

E_2) Găsim valorile funcției (imaginea funcției)

E_3) $f'(x)=\left(\frac{e^x}{x+2}\right)'=\frac{(e^x)'(x+2)-e^x(x+2)'}{(x+2)^2}$

$=\frac{e^x(x+2)-e^x\cdot 1}{(x+2)^2}=\frac{e^x(x+2-1)}{(x+2)^2}=\frac{e^x(x+1)}{(x+2)^2}$

E_4) $f'(x)=0\Rightarrow e^x(x+1)=0\Rightarrow$ $\begin{cases} e^x=0 \text{ fals} \\ \text{sau} \\ x+1=0\Rightarrow x=-1\notin[0;1] \end{cases}$

E_5)

x	$[0$		$1]$
f'		$+++$	
f	$\frac{1}{2}$	$\nearrow$	$\frac{e}{3}$

$f(0)=\frac{e^0}{0+2}=\frac{1}{2}$; $f(1)=\frac{e^1}{1+2}=\frac{e}{3}$

$\frac{1}{2}\le f(x)\le\frac{e}{3}\Rightarrow\frac{3}{e}\le\frac{1}{f(x)}\le$

Găsiți $a \in \mathbb{R}$ pentru care $\operatorname{Im} f = [-3, \infty)$ unde $f(x) = x^2 + ax + 1$

E_1) $\operatorname{Im} f$ este reuniunea tuturor intervalelor pe care f este strict monotonă

E_2) $f'(x) = 2x + a$

$f'(x) = 0 \Rightarrow 2x + a = 0 \Rightarrow x = -\frac{a}{2}$

E_3)

x	$-\infty$		$-\frac{a}{2}$		∞
f'	$-$	$-$	0	$+$	$+$
f	∞	$\searrow$	$\frac{-a^2+4}{4}$	$\nearrow$	∞

E_4) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} (x^2 + ax + 1) = \lim\limits_{x \to -\infty} x^2 = \infty$

$f(-\frac{a}{2}) = (-\frac{a}{2})^2 + a \cdot (-\frac{a}{2}) + 1 =$

$= \frac{a^2}{4} - \frac{2a^2}{2} + 1 = \frac{a^2 - 2a^2 + 4}{4} = \frac{-a^2+4}{4}$

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} (x^2 + ax + 1) = \lim\limits_{x \to \infty} x^2 = \infty$

E_5) Din tabel $\Rightarrow \operatorname{Im} f = \left[\frac{-a^2+4}{4}, \infty\right) \cup \left[\frac{-a^2+4}{4}, \infty\right)$

$= \left[\frac{-a^2+4}{4}, \infty\right)$

E_6) Dar, $\operatorname{Im} f = [-3, \infty)$ din enunț

$\Rightarrow \frac{-a^2+4}{4} = -3 \Rightarrow$ ~~$-a^2+4 = -12 \Rightarrow$~~

$a^2 + 4 = -12 \Rightarrow -a^2 = -16 \Rightarrow a^2 = 16 \Rightarrow a = \pm 4$

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