CALCULUL PUTERILOR UNEI MATRICI

Calculați A^n, $n \geq 2$ în fiecare din situațiile:

1) $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

2) $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3) $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

4) $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

5) $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

6) $A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

9) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

10) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

11) $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

12) $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

13) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ NEWTON

14) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

15) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ inducție

16) $A = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$, se cere B^n unde $B = I_2 + A$

17) $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, se cere $(I_3 + A)^n$

18) $A(\alpha) = \begin{pmatrix} 1 & \alpha & 2\alpha^2 + 2\alpha \\ 0 & 1 & 4\alpha \\ 0 & 0 & 1 \end{pmatrix}$ se cere $(A(\alpha))^n$,

19) $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, se știe că $A^{n+1} = 3A^n$; se cere $S = A + A^2 + \dots + A^n$

20) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, se cere $A^n - A^{n-2} = A^2 - I_3$, $n \geq 3$

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Fie $A=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, Se cere A^n, $n\in$

NEWTON

$E_1)$ $B=A-I_3 \Rightarrow A=I_3+B$

$\Rightarrow B=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$E_2)$ Calc. $B^2=\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow B^2=O_3$, $\forall n\geq 2$

$E_3)$ $A^n=(I_3+B)^n=C_n^0 I_3^{n-0}B^0+C_n^1 I_3^{n-1}B+$
$+C_n^2 I_3^{n-2}B^2+\dots+C_n^n I_3^{n-n}B^n$

$\Rightarrow A^n=C_n^0 I_3^n B^0+C_n^1 I_3^{n-1}B+C_n^2 I_3^{n-2}O_3+\dots+O_3$

$A^n=\frac{n!}{0!(n-0)!}\cdot I_3\cdot I_3+\frac{n!}{1!(n-1)!}I_3B+O_3$

$E_4)$ $A^n=1\cdot I_3+nB+O_3$

$A^n=I_3+nB\Rightarrow A^n=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+n\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow A^n=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+\begin{pmatrix} 0 & 0 & -2n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Obs: S-a folosit $I_3^\Delta=I_3$, $B^0=I_3$

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Fie $A=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}$, să se A^n, $n\geq 2$

E_1) Calc. A^2, A^3, ... și se observă regula

E_2) folosim NEWTON: $A=I_3+B \Rightarrow$

$$B=A-I_3=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$$

E_3) $B^2=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$

$$B^3=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O_3$$

E_4) $n\leq (I_3+B)^n=C_n^0 I_3^{n-0}B^0+C_n^1 I_3^{n-1}B+$

$+C_n^2 I_3^{n-2}B^2+C_n^3 I_3^{n-3}B^3+\dots+C_n^n I_3^0 B^n$

E_5) Dar, $B^3=O_3 \Rightarrow B^p=O_3, \forall p\geq 3$

E_6) $A^n=C_n^0 I_3+C_n^1 B+C_n^2 B^2+O_3$

E_7) Dar, $C_n^0=1$, $C_n^1=n$, $C_n^2=\frac{n(n-1)}{2}$

$\Rightarrow A^n=I_3+nB+\frac{n(n-1)}{2}B^2=$

$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\n&0&0\\0&n&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\\frac{n(n-1)}{2}&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\n&1&0\\\frac{n(n-1)}{2}&n&1\end{pmatrix}$

Prof. Ovidiu Bădescu

Fie $A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, calculați A^{2014}

E1) $A^2=A\cdot A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}=-I_2$

E2) $A^3=A^2\cdot A=-I_2\cdot A=-A$

E3) $A^4=A^3\cdot A=-A\cdot A=-A^2=-(-I_2)=I_2$

E4) $A^{2014}=?$

$2014 \mid 4$
$20 \quad 503$
$==14$
12
$==2$

$\Rightarrow A^{2014}=A^{4\cdot 503+2}=$

$=(A^4)^{503}\cdot A^2=I_2\cdot A^2$

$=A^2=-I_2$

Prof. Ovidiu Badescu

Fie $A=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, calc. A^n observând regula și arătați prin inducție

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$E_1)$ $A^2=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (*)

$E_2)$ $A^3=A^2\cdot A=\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_3)$ Dem. că $A^n=\begin{pmatrix} 1 & 0 & -2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, cu inducție

$E_4)$ I $P(2)$: $A^2=\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ adev. din (*)

II Pp. $P(k)$ adev. $\overset{II}{\Longrightarrow}$ $P(k+1)$ adev. unde

$P(k)$: $A^k=\begin{pmatrix} 1 & 0 & -2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P(k+1)$: $A^{k+1}=\begin{pmatrix} 1 & 0 & -2k-2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Știm că $P(k)$ adev. $\Rightarrow A^k=\begin{pmatrix} 1 & 0 & -2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Big|\cdot A$

$\Rightarrow A^{k+1}=\begin{pmatrix} 1 & 0 & -2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -2-2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_5)\Rightarrow P(k+1)$ adev. $\Rightarrow P(n)$ adev. $\forall n\geq 2\Rightarrow$

$A^n=\begin{pmatrix} 1 & 0 & -2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Prof. Ovidiu Bădescu

TDG
II 9b) $A=\begin{pmatrix}2&2\\-2&-2\end{pmatrix}$, se cere $B^n=?$ unde II 9b

$B=I_2+A$

E1) $B^n=(I_2+A)^n=C_n^0 I_2^n A^0+C_n^1 I_2^{n-1}A+$
$\dots+C_n^n I_2^{n-n}A^n$

E2) $A^2=\begin{pmatrix}2&2\\-2&-2\end{pmatrix}\begin{pmatrix}2&2\\-2&-2\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=O_2$

$\Rightarrow A^p=O_2,\ \forall p\geq 2$

E3) $B^n=C_n^0 I_2\cdot I_2+C_n^1 I_2\cdot A+O_2$

$B^n=C_n^0 I_2+C_n^1 A$

E4) $C_n^0=\frac{n!}{0!(n-0)!}=1$

$C_n^1=\frac{n!}{1!(n-1)!}=\frac{(n-1)!\cdot n}{(n-1)!}=n$

E5) $B^n=1\cdot I_2+nA=\begin{pmatrix}1&0\\0&1\end{pmatrix}+n\begin{pmatrix}2&2\\-2&-2\end{pmatrix}$
$=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}2n&2n\\-2n&-2n\end{pmatrix}=\begin{pmatrix}2n+1&2n\\-2n&-2n+1\end{pmatrix}$

Obs.: Dacă x folosește binomul lui Newton
o necesară inducția

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T 8
II 1c) Fie $A=\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ cu

T 125
II 1c)

Se cere: $(I_3+A)^n$

E1) $S=(I_3+A)^n = C_n^0 I_3^n A^0 + C_n^1 I_3^{n-1} A + C_n^2 I_3^{n-2} A^2 + \dots + C_n^n I_3^{n-n} A^n$

E2) $S=C_n^0 I_3 + C_n^1 A + C_n^2 A^2 + \dots + C_n^n A^n$

E3) Dar $A^k = 3^{k-1} A \Rightarrow$

$S=C_n^0 I_3 + C_n^1 A + C_n^2 3A + \dots + C_n^n 3^{n-1} \cdot A$

$S=C_n^0 I_3 + A\left(C_n^1 + C_n^2 3 + \dots + C_n^n 3^{n-1}\right)$

E4) $N=C_n^1 + C_n^2 3 + \dots + C_n^n 3^{n-1}$, se găsește

Q NEWTON $\Rightarrow N=\frac{1}{3}\left(C_n^1 3 + C_n^2 3^2 + \dots + C_n^n 3^n\right)$

$=\frac{1}{3}\left(\underbrace{C_n^0 + C_n^1 3 + \dots + C_n^n 3^n}_{(1+3)^n} - \underbrace{C_n^0}_{1}\right)$

$=\frac{1}{3}\left((1+3)^n - 1\right) = \frac{4^n-1}{3}$

$= \dots = I_2 + \frac{4^n-1}{3} A$

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V28
II 2(1) Fie $A(\alpha) = \begin{pmatrix} 1 & \alpha & 2\alpha^2+2\alpha \\ 0 & 1 & 4\alpha \\ 0 & 0 & 1 \end{pmatrix}$, $\alpha \in \mathbb{R}$

$(A(\alpha))^n$

E1) Calc. $A(\alpha)\cdot A(\beta) \overset{\text{calcul}}{=\!=} A(\alpha+\beta)$

E2) $(A(\alpha))^2 = A(\alpha)\cdot A(\alpha) = A(2\alpha)$

Obs. că $(A(\alpha))^n = A(n\alpha)$, $\forall n \geq 2$

E3) inducție

I $P(2): (A(\alpha))^2 = A(2\alpha)$ adev.

II Fie n. $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

$P(k): (A(\alpha))^k = A(k\alpha)$

$P(k+1): (A(\alpha))^{k+1} = A((k+1)\alpha)$

Știm că $(A(\alpha))^k = A(k\alpha) \mid \cdot A(\alpha)$

$(A(\alpha))^{k+1} = A(k\alpha)\cdot A(\alpha) \overset{E1)}{=\!=}$

$= A(k\alpha+\alpha) = A((k+1)\alpha)$

$\Rightarrow P(k+1)$ adev. $\Rightarrow P(n)$ adev., $\forall n \geq 2$

Prof. Ovidiu Bădescu

+8

II 1b) $A=\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, știm că $A^{n+1}=3A^n$

T 125
II 1b)

Calculați $S_n = A + A^2 + \dots + A^n$

E_1) calculez A^n în funcție de A

din $A^{n+1}=3A^n \Rightarrow A^2 = 3A$

$A^3 = 3A^2 = 3 \cdot 3A = 3^2 A$

Obs. că $A^n = 3^{n-1}A$, $\forall n \geq 1$

E_2) ind.: I $P(1)$: $A^1 = 3^0 \cdot A \Rightarrow A = A$ adev.

II Fie $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

$P(k)$: $A^k = 3^{k-1}A$

$P(k+1)$: $A^{k+1} = 3^k A$

Știm că $P(k)$ adev. $\Rightarrow A^k = 3^{k-1}A \,|\cdot A$

$\Rightarrow A^{k+1} = 3^{k-1}A^2$

Dar, $A^2 = 3A$ $\Rightarrow A^{k+1} = 3^k A$

$\Rightarrow P(k+1)$ adev. $\Rightarrow P(n)$ adev., $\forall n \geq 1$

E_3) $A + A^2 + \dots + A^n = A + 3A + 3^2A + \dots + 3^{n-1}A$

$= (1 + 3 + 3^2 + \dots + 3^{n-1})A = \frac{1(3^n - 1)}{3-1} A = \frac{3^n - 1}{2} A$

1 + q + q^2 + ... + q^n = (q^{n+1} - 1)/(q - 1)

(284)

Fie $A=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}$, arătați că $A^n-A^{n-2}=A^2-I_3$, $\forall n\geq 3$

E_1) Inducție, $P(n)$: $A^n-A^{n-2}=A^2-I_3$

E_2) I $P(3)$: $A^3-A=A^2-I_3$

$A^2=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}$ și $A^3=A^2\cdot A=\begin{pmatrix}1&2&1\\0&0&1\\0&1&0\end{pmatrix}$

și evident $A^3-A=A^2-I_3$ (calcul)

E_3) II $P(k)$ adev. $\overset{(?)}{\Rightarrow}$ $P(k+1)$ adev. unde

$P(k)$: $A^k-A^{k-2}=A^2-I_3$

$P(k+1)$: $A^{k+1}-A^{k-1}=A^2-I_3$

Știm că $P(k)$ adev $\Rightarrow A^k-A^{k-2}=A^2-I_3\,/\cdot A$

$\Rightarrow A^{k+1}-A^{k-1}=A^3-A$

Dar, din et verificării $\Rightarrow A^3-A=A^2-I_3\Rightarrow$

$A^3=A^2+A-I_3\Rightarrow A^{k+1}-A^{k-1}=A^2+A-I_3-A=$

$=A^2-I_3$

$\Rightarrow P(k+1)$ adev $\Rightarrow P(n)$ adev., $\forall n\geq 3$

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