PROBLEME NON STANDARD CU MATRICI

1) Fie $A=\begin{pmatrix}-1 & 2\\ -3 & 4\end{pmatrix}$ și $B=\begin{pmatrix}5 & -4\\ 3 & -2\end{pmatrix}$

Arătați că, $\forall n\geq 1$ avem

$A^n-B^n=(2^n-1)(A-B)$

2) Fie $A=\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$, se cere

A^n, $n\geq 1$

3) se cere $A^n=?$, $n\geq 2$ dacă

$A=\begin{pmatrix}2 & 0 & 1\\ 0 & -3 & 0\\ 1 & 0 & 2\end{pmatrix}$

4) se are $A^n=?$, $n\geq 2$ dacă

$A=\begin{pmatrix}1+\sqrt{3} & 1-\sqrt{3}\\ \sqrt{3}-1 & \sqrt{3}+1\end{pmatrix}$

5) Fie $A=\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$.

se cere A^n

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Fie $A=\begin{pmatrix}-1 & 2\\ -3 & 4\end{pmatrix}$ și $B=\begin{pmatrix}5 & -4\\ 3 & -2\end{pmatrix}$

Arătați că $A^n-B^n=(2^n-1)(A-B)$, $\forall n\geq 1$

E_1) Folosim ec. C-H: $A^2-(\mathrm{Tr}A)\cdot A+\det A\cdot I_2=O_2$

$\Rightarrow A^2-3A+2I_2=O_2 \Rightarrow A^2=3A-2I_2 \mid \cdot A \Rightarrow$

$A^3=3A^2-2A \Rightarrow A^3=3(3A-2I_2)-2A=7A-6I_2 \mid \cdot A$

E_2) $A^4=7A^2-6A=7(3A-2I_2)-6A=21A-14I_2-6A$

$\Rightarrow A^4=15A-14I_2$

Dem. că $A^n=(2^n-1)A-(2^n-2)I_2$, $\forall n\geq 1$, inducție

E_3) I $P(1)$: $A=(2-1)A-0\cdot I_2$ adev.

II Pp. $P(k)$ adev $\overset{(?)}{\Rightarrow} P(k+1)$ adev unde

$P(k)$: $A^k=(2^k-1)A-(2^k-2)I_2$

$P(k+1)$: $A^{k+1}=(2^{k+1}-1)A-(2^{k+1}-2)I_2$

E_4) Din $P(k)$ adev $\Rightarrow A^k=(2^k-1)A-(2^k-2)I_2 \mid \cdot A$

$\Rightarrow A^{k+1}=(2^k-1)A^2-(2^k-2)A=(2^k-1)(3A-2I_2)-(2^k-2)A$

$\Rightarrow A^{k+1}=(3\cdot 2^k-3-2^k+2)A-(2^{k+1}-2)I_2 \Rightarrow$

$\Rightarrow A^{k+1}=(2\cdot 2^k-1)A-(2^{k+1}-2)I_2=(2^{k+1}-1)A-(2^{k+1}-2)I_2$

$\Rightarrow P(k+1)$ adev $\Rightarrow P(n)$ adev., $\forall n\geq 1$

E_5) Analog, B verif ec. C-H $\Rightarrow B^2-3A+2I_2=O_2$

$\Rightarrow B^2=3B-2I_2 \overset{\text{inducție}}{\Longrightarrow} B^n=(2^n-1)B-(2^n-2)I_2$

E_6) $A^n-B^n=(2^n-1)(A-B)-0 \Rightarrow$ e.e.d.

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Calculați A^n, $n \geq 2$ pt $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$E_1)$ $A^2 = A \cdot A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 9 & 0 \\ 4 & 0 & 5 \end{pmatrix}$ nu mai avem regula

$E_2)$ $A = I_3 + B$ unde $B = A - I_3 \Rightarrow B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$E_3)$ $B^2 = B \cdot B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 16 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, $B^3 = \begin{pmatrix} 4 & 0 & 4 \\ 0 & -64 & 0 \\ 4 & 0 & 4 \end{pmatrix}$

și deducem că $B^n = \begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & (-4)^n & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix}$, $\forall n \geq 1$ și se arată prin inducție că e adevărată

$E_4)$ $A^n = (I_3 + B)^n = C_n^0 I_3^n B^0 + C_n^1 I_3^{n-1} B + C_n^2 I_3^{n-2} B^2 + \dots + C_n^n I_3^{n-n} B^n$

$= C_n^0 I_3 + C_n^1 B + C_n^2 B^2 + \dots + C_n^n B^n =$

$= C_n^0 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + C_n^1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{pmatrix} + C_n^2 \begin{pmatrix} 2 & 0 & 2 \\ 0 & 16 & 0 \\ 2 & 0 & 2 \end{pmatrix} + \dots + C_n^n \begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & (-4)^n & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix}$

$\Rightarrow A^n = \begin{pmatrix} C_n^0 + C_n^1 + C_n^2 \cdot 2 + \dots + C_n^n 2^{n-1} & 0 & 0 + C_n^1 + C_n^2 \cdot 2 + \dots + C_n^n 2^{n-1} \\ 0 & C_n^0 + C_n^1(-4) + \dots + C_n^n (-4)^n & 0 \\ 0 + C_n^1 + C_n^2 \cdot 2 + \dots + C_n^n 2^{n-1} & 0 & C_n^0 + C_n^1 + C_n^2 \cdot 2 + \dots + C_n^n 2^{n-1} \end{pmatrix}$

$E_5)$ Fie $S_1 = C_n^1 + C_n^2 \cdot 2 + \dots + C_n^n 2^{n-1} \mid \cdot 2 \Rightarrow 2S_1 = C_n^1 2 + C_n^2 2^2 + \dots + C_n^n 2^n \mid + C_n^0$

$\Rightarrow C_n^0 + 2S_1 = C_n^0 + C_n^1 2 + C_n^2 2^2 + \dots + C_n^n 2^n \Rightarrow 1 + 2S_1 = (1+2)^n \Rightarrow$

$1 + 2S_1 = 3^n \Rightarrow S_1 = \frac{3^n - 1}{2} \Rightarrow$ term $a_{11} = 1 + \frac{3^n - 1}{2} = \boxed{\frac{3^n + 1}{2}}$

$E_6)$ Fie $S_2 = C_n^0 + C_n^1 \cdot (-4) + C_n^2 (-4)^2 + \dots + C_n^n (-4)^n \Rightarrow S_2 = (1-4)^n = (-3)^n \Rightarrow a_{22} = (-3)^n$

$E_7)$ a_{13} este $S_1 = \frac{3^n - 1}{2} \Rightarrow$

$$A^n = \begin{pmatrix} \frac{3^n + 1}{2} & 0 & \frac{3^n - 1}{2} \\ 0 & (-3)^n & 0 \\ \frac{3^n - 1}{2} & 0 & \frac{3^n + 1}{2} \end{pmatrix}$$

Calculați A^n, $n \geq 2$ și $A = \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix}$

E_1) $A = \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ -(1-\sqrt{3}) & 1+\sqrt{3} \end{pmatrix} \Rightarrow A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, încercăm să o scriem trigonometric sub forma $A = \alpha \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$

unde $\alpha = \sqrt{a^2+b^2} \Rightarrow \alpha = \sqrt{(1+\sqrt{3})^2+(1-\sqrt{3})^2} = \sqrt{1+2\sqrt{3}+3+1-2\sqrt{3}+3}$

$\Rightarrow \alpha = \sqrt{8} = 2\sqrt{2} \Rightarrow A = 2\sqrt{2} \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} \\ -\frac{1-\sqrt{3}}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$

$\Rightarrow A = 2\sqrt{2} \begin{pmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ -\frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{2}+\sqrt{6}}{4} \end{pmatrix}$

E_2) ar trebui $\cos\varphi = \frac{\sqrt{2}+\sqrt{2}\cdot\sqrt{3}}{4} \Rightarrow \cos\varphi = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}$,

dar $\cos\varphi = \cos x \cos y + \sin x \sin y \Rightarrow \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$\Rightarrow \varphi = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \Rightarrow A = 2\sqrt{2} \begin{pmatrix} \cos\frac{\pi}{12} & \sin\frac{\pi}{12} \\ -\sin\frac{\pi}{12} & \cos\frac{\pi}{12} \end{pmatrix}$

E_3) A are forma $A = \alpha \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, se demonstrează prin inducție (dar aici nu mai am loc) că $A^n = \alpha^n \begin{pmatrix} \cos n\varphi & \sin n\varphi \\ -\sin n\varphi & \cos n\varphi \end{pmatrix}$

$\Rightarrow A^n = (2\sqrt{2})^n \begin{pmatrix} \cos\frac{n\cdot\pi}{12} & \sin\frac{n\pi}{12} \\ -\sin\frac{n\pi}{12} & \cos\frac{n\pi}{12} \end{pmatrix}$

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Fie $A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$, calculați A^n, $\forall n\geq 1$

E_1) încercăm $A^2=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}=\begin{pmatrix}6&5&5\\5&6&5\\5&5&6\end{pmatrix}$ și calculăm $A^3 \Rightarrow$ nu găsim regulă

E_2) $A=I_3+B$ unde $B=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

E_3) $B^2=B\cdot B=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}=\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix}=3B$

$B^3=B^2\cdot B=3B\cdot B=3B^2=3\cdot 3B=3^2B$, arăt. $B^n=3^{n-1}B$

E_4) $A^n=(I_3+B)^n=C_n^0 I_3^n B^0+C_n^1 I_3^{n-1}B+C_n^2 I_3^{n-2}B^2+\dots+C_n^n I_3^0 B^n$

Folosind $I_3^k=I_3$, $B^0=I_3$ și că $B^k=3^{k-1}B$

E_5) $A^n=C_n^0 I_3+C_n^1 B+C_n^2 3B+\dots+C_n^n 3^{n-1}B$

$A^n=C_n^0 I_3+B\left(C_n^1+C_n^2 3+\dots+C_n^n 3^{n-1}\right)$

E_6) Aplicăm forma Newton în paranteză $\Rightarrow$

$A^n=C_n^0 I_3+B\left(C_n^1 3+C_n^2 3^2+\dots+C_n^n 3^n\right)\cdot\frac{1}{3}$

$A^n=C_n^0 I_3+B\left(\underbrace{C_n^0+C_n^1 3+C_n^2 3^2+\dots+C_n^n 3^n}_{(1+3)^n}-\underbrace{C_n^0}_{=1}\right)\cdot\frac{1}{3}$

$A^n=C_n^0 I_3+B\cdot\frac{4^n-1}{3}\Rightarrow A^n=1\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{4^n-1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$\Rightarrow A^n=\begin{pmatrix}1+\frac{4^n-1}{3}&\frac{4^n-1}{3}&\frac{4^n-1}{3}\\ \dots&\dots&\dots\\ \dots&\dots&\dots\end{pmatrix}=\begin{pmatrix}\frac{4^n+2}{3}&\frac{4^n-1}{3}&\frac{4^n-1}{3}\\ \frac{4^n-1}{3}&\frac{4^n+2}{3}&\frac{4^n-1}{3}\\ \frac{4^n-1}{3}&\frac{4^n-1}{3}&\frac{4^n+2}{3}\end{pmatrix}$

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