DEDUCEREA RELAȚIILOR GEOMETRICE DIN TEOREMA COSINUSULUI

IX.70

Fie $\triangle ABC$ cu laturile $BC=a$, $AB=c$, $AC=b$, arătați relațiile:

1) $a^2=b^2+c^2-2bc\cos A$ (T. cos)

2) $b^2=a^2+c^2-2ac\cos B$ (T. cos)

3) $c^2=a^2+b^2-2ab\cos A$ (T. cos)

4) $AB^2\cdot DC-AD^2\cdot BC+AC^2\cdot BD=BD\cdot DC\cdot BC$, $D\in(BC)$

5) $BC^2\cdot AE-BE^2\cdot AC+BA^2\cdot CE=AE\cdot CE\cdot AC$, $E\in(AC)$

6) $CA^2\cdot BF-CF^2\cdot AB+CB^2\cdot AF=AF\cdot BF\cdot AB$, $F\in(AB)$

7) $m_a^2=\dfrac{2(b^2+c^2)-a^2}{4}$ (lung medianei)

8) $m_b^2=\dfrac{2(a^2+c^2)-b^2}{4}$ (lung medianei)

9) $m_c^2=\dfrac{2(a^2+b^2)-c^2}{4}$ (lung medianei)

10) $l_a=\dfrac{2}{b+c}\sqrt{bcp(p-a)}$

11) $l_b=\dfrac{2}{a+c}\sqrt{acp(p-b)}$

12) $l_c=\dfrac{2}{a+b}\sqrt{abp(p-c)}$

13) $MA^2+MB^2+MC^2=3MG^2+GA^2+GB^2+GC^2$, $\forall M\in$ plan (relația LEIBNIZ)

14) $\cos\dfrac{A}{2}=\sqrt{\dfrac{p(p-a)}{bc}}$, 15) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(p-b)(p-c)}{bc}}$

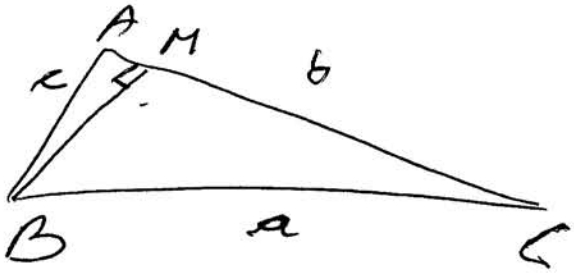
16) $\cos\dfrac{B}{2}=\sqrt{\dfrac{p(p-b)}{ac}}$, 17) $\sin\dfrac{B}{2}=\sqrt{\dfrac{(p-a)(p-c)}{ac}}$

18) $\cos\dfrac{C}{2}=\sqrt{\dfrac{p(p-c)}{ab}}$, 19) $\sin\dfrac{C}{2}=\sqrt{\dfrac{(p-a)(p-b)}{ab}}$

Prof. Dumitru Bădescu

TEOREMA COSINUSULUI

$\vec{I_m}$ $\forall \Delta ABC \Rightarrow a^2 = b^2 + c^2 - 2bc \cdot \cos A$



E_1) Fie $BM \perp AC$ (se putea $CN \perp AB$ dar nu se putea $AP \perp BC$ pt că împărțeam latura a cu trebuie)

E_2) în ΔBMC, T. Pit $\Rightarrow a^2 = BM^2 + MC^2$

E_3) în ΔABM, T. Pitag $\Rightarrow BM^2 = c^2 - AM^2$

înloc în E_2) $\Rightarrow$

$$a^2 = c^2 - AM^2 + MC^2$$

E_4) ne trebuie cos A, trebuie să avem laturi

din $\Delta ABM \Rightarrow MC = b - AM \overset{E_3)}{\Rightarrow}$

$$a^2 = c^2 - AM^2 + (b - AM)^2$$

$$a^2 = c^2 - \cancel{AM^2} + b^2 - 2bAM + \cancel{AM^2}$$

$$a^2 = b^2 + c^2 - 2b \cdot AM \quad (*)$$

E_5) ne încurcă AM, trebuie cos A

$\Rightarrow$ în $\Delta BAM \Rightarrow \cos A = \frac{AM}{c} \Rightarrow AM = c \cdot \cos A$

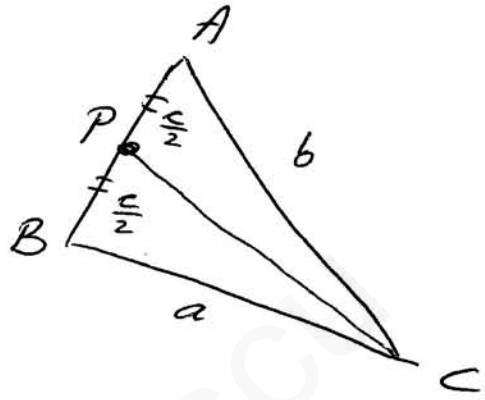
E_6) înlocuim în $(*) \Rightarrow$

$$\boxed{a^2 = b^2 + c^2 - 2b \cdot c \cdot \cos A}$$

Prof. Ovidiu Badescu

Arătați că: $m_c^2 = \frac{2(a^2+b^2)-c^2}{4}$ unde

m_c este mediana din C pt $\triangle ABC$



E_1) din relația lui SYLVESTER

$\Rightarrow CA^2 \cdot PB - CP^2 \cdot AB + CB^2 \cdot AP =$

$= AP \cdot PB \cdot AB$

E_2) $b^2 \cdot \frac{c}{2} - m_c^2 \cdot c + a^2 \cdot \frac{c}{2} = \frac{c}{2} \cdot \frac{c}{2} \cdot c \,/ : c$

$\Rightarrow \frac{b^2}{2} - m_c^2 + \frac{a^2}{2} = \frac{c^2}{4}$

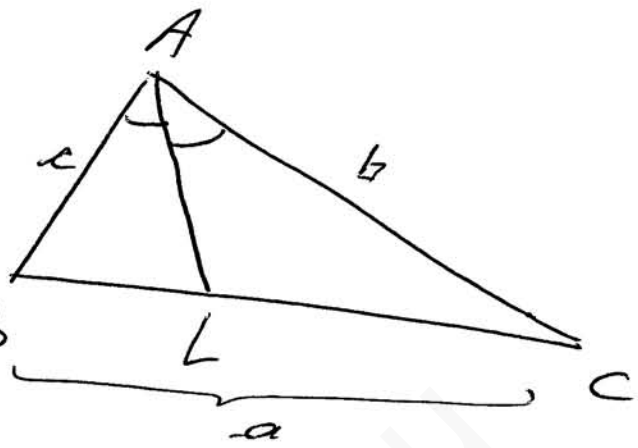
$\Rightarrow m_c^2 = \frac{b^2}{2} + \frac{a^2}{2} - \frac{c^2}{4}$

$\Rightarrow m_c^2 = \frac{2(a^2+b^2)-c^2}{4}$

Prof. Ovidiu Badescu

Fie $\triangle ABC$ și l_a lungimea bisectoarei din A, se cere

$$l_a = \frac{2}{b+c}\sqrt{bcp(p-a)}$$



E_1) $L \in (BC)$, din relația lui STEWART avem

$$\boxed{AB^2 \cdot LC - AL^2 \cdot BC + AC^2 \cdot BL = BL \cdot BC \cdot CL}$$

E_2) din T. bisectoarei $\Rightarrow$

$$\frac{AB}{AC} = \frac{BL}{LC} \Rightarrow \frac{c}{b} = \frac{BL}{LC}$$

E_3) folosind proporții derivate, $\frac{BL}{LC} = \frac{c}{b} \Rightarrow \frac{BL}{BL+LC} = \frac{c}{c+b} \Rightarrow$

$$\Rightarrow \frac{BL}{BC} = \frac{c}{c+b} \Rightarrow \frac{BL}{a} = \frac{c}{c+b} \Rightarrow \boxed{BL = \frac{ac}{b+c}}$$

Analog, $\frac{BL}{LC} = \frac{c}{b} \Rightarrow \frac{BL+LC}{LC} = \frac{c+b}{b} \Rightarrow \frac{BC}{LC} = \frac{c+b}{b} \Rightarrow$

$$\Rightarrow \frac{a}{LC} = \frac{c+b}{b} \Rightarrow \boxed{CL = \frac{ab}{b+c}}$$

E_4) înlocuim în relația de la E_1) $\Rightarrow$

$$c^2 \cdot \frac{ab}{b+c} - AL^2 \cdot a + b^2 \cdot \frac{ac}{b+c} = \frac{ac}{b+c} \cdot a \cdot \frac{ab}{b+c} \quad \Big| : a$$

$$\Rightarrow \frac{bc^2}{b+c} - AL^2 + \frac{cb^2}{b+c} = \frac{a^2bc}{(b+c)^2}$$

$$\Rightarrow AL^2 = \frac{bc^2}{b+c} + \frac{cb^2}{b+c} - \frac{a^2bc}{(b+c)^2}$$

$$AL^2 = \frac{bc}{b+c}\left(c+b-\frac{a^2}{b+c}\right) \Rightarrow$$

$$AL^2 = \frac{bc}{b+c} \cdot \frac{(b+c)^2 - a^2}{b+c} = \frac{bc}{(b+c)^2}\left((b+c-a)(b+c+a)\right)$$

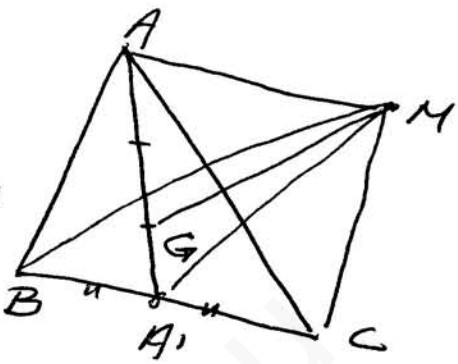
E_5) $AL^2 = \frac{bc}{(b+c)^2}(a+b+c-2a)(a+b+c) = \frac{bc(2p-2a)(2p)}{(b+c)^2}$

E_6) $\boxed{l_a = \frac{2}{b+c}\sqrt{bcp(p-a)}}$

Prof. Ovidiu Bădescu

Fie ΔABC cu t/M din plan și G centru de greutate

$\Rightarrow MA^2+MB^2+MC^2=3MG^2+GA^2+GB^2+GC^2$ (LEIBNITZ)



E_1) în $\Delta MAA'$ aplicăm STEWART

$\Rightarrow MA^2\cdot GA'-MG^2\cdot AA'+MA'^2\cdot GA=AG\cdot GA'\cdot AA'$

E_2) dar, G centru de greutate $\Rightarrow$

$GA'=\frac{1}{3}AA'$, $GA=\frac{2}{3}AA'\Rightarrow$

$MA^2\cdot\frac{1}{3}AA'-MG^2\cdot AA'+MA'^2\cdot\frac{2}{3}AA'=AG\cdot GA'\cdot AA' \mid :AA'$

$\Rightarrow MA^2\cdot\frac{1}{3}-MG^2+MA'^2\cdot\frac{2}{3}=AG\cdot GA'$

$$\Rightarrow \boxed{MG^2=\frac{1}{3}MA^2+\frac{2}{3}MA'^2-\frac{2}{9}AA'^2} \quad (1)$$

E_3) dar, MA' mediană în $\Delta MBC\Rightarrow$

$$\boxed{MA'^2=\frac{2(MB^2+MC^2)-BC^2}{4}} \quad (2)$$

și AA' mediană în $\Delta ABC\Rightarrow$ $\boxed{AA'^2=\frac{2(AB^2+AC^2)-BC^2}{4}}$

E_4) înlocuim în (1)

$\Rightarrow MG^2=\frac{1}{3}MA^2+\frac{2}{3}\cdot\frac{2(MB^2+MC^2)-BC^2}{4}-\frac{2}{9}\cdot\frac{2(AB^2+AC^2)-BC^2}{4}$

$$\Rightarrow \boxed{MG^2=\frac{1}{3}\left(MA^2+MB^2+MC^2\right)-\frac{1}{9}\left(AB^2+AC^2+BC^2\right)} \quad (1)$$

E_5) în formulă apare $GA^2+GB^2+GC^2=\frac{4}{9}\left(m_a^2+m_b^2+m_c^2\right)$

$$=\frac{4}{9}\left(\frac{2(b^2+c^2)-a^2}{4}+\frac{2(a^2+c^2)-b^2}{4}+\frac{2(a^2+b^2)-c^2}{4}\right)=\frac{4}{9}\cdot\frac{3(a^2+b^2+c^2)}{4}$$

$\Rightarrow a^2+b^2+c^2=3(GA^2+GB^2+GC^2)\Rightarrow MG^2=\frac{1}{3}(MA^2+MB^2+MC^2)-\frac{1}{9}\cdot 3(GA^2+GB^2+GC^2)$

$\Rightarrow \boxed{MG^2=MA^2+MB^2+MC^2-(GA^2+GB^2+GC^2)}$

Prof. Ovidiu Bădoiescu

Arătați că: $\cos\frac{A}{2}=\sqrt{\frac{p(p-a)}{bc}}$

E_1) $\exists$ formula $\cos^2 x=\frac{1+\cos 2x}{2} \Rightarrow$

$$\cos^2\frac{A}{2}=\frac{1+\cos A}{2} \quad (1)$$

E_2) din T. cos $\Rightarrow \cos A=\frac{b^2+c^2-a^2}{2bc}$

$$\Rightarrow \cos^2\frac{A}{2}=\frac{1+\frac{b^2+c^2-a^2}{2bc}}{2}=\frac{\frac{2bc+b^2+c^2-a^2}{2bc}}{2}=$$

$$=\frac{(b+c)^2-a^2}{4bc}=\frac{(b+c-a)(b+c+a)}{4bc}$$

E_3) dar, $a+b+c=2p \Rightarrow$

$$\cos^2\frac{A}{2}=\frac{(b+c+a-2a)(b+c+a)}{4bc}=\frac{(2p-2a)\cdot 2p}{4bc}$$

$$\cos^2\frac{A}{2}=\frac{4(p-a)p}{4bc} \Rightarrow \cos\frac{A}{2}=\pm\sqrt{\frac{p(p-a)}{bc}}$$

E_4) A unghi al triunghiului $\Rightarrow A\in(0;\pi)$

$$\Rightarrow \frac{A}{2}\in\left(0;\frac{\pi}{2}\right) \Rightarrow \frac{A}{2}\in C_1 I \Rightarrow \cos\frac{A}{2}>0$$

$$\Rightarrow \boxed{\cos\frac{A}{2}=\sqrt{\frac{p(p-a)}{bc}}}$$

Prof Ovidiu Bădescu

Arătați că: $\sin\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}$

E_1) există formula $\sin^2 x=\frac{1-\cos 2x}{2}\Rightarrow$

$$\sin^2\frac{A}{2}=\frac{1-\cos A}{2}\quad(1)$$

E_2) din T cos $\Rightarrow a^2=b^2+c^2-2bc\cos A\Rightarrow\cos A=\frac{b^2+c^2-a^2}{2bc}$

E_3) înlocuim în (1) $\sin^2\frac{A}{2}=\frac{1-\frac{b^2+c^2-a^2}{2bc}}{2}=\frac{2bc-b^2-c^2+a^2}{4bc}$

$$\sin^2\frac{A}{2}=\frac{a^2-(b^2-2bc+c^2)}{4bc}\Rightarrow\sin^2\frac{A}{2}=\frac{a^2-(b-c)^2}{4bc}$$

E_4) $\sin^2\frac{A}{2}=\frac{(a-b+c)(a+b-c)}{4bc}\quad(2)$

E_5) dar, $\frac{a+b+c}{2}=p\Rightarrow a+b+c=2p\mid-2b$

$\Rightarrow a-b+c=2p-2b=2(p-b)$

din $a+b+c=2p\mid-2c\Rightarrow a+b-2c=2p-2c$

E_6) înlocuim în (2) $\Rightarrow\sin^2\frac{A}{2}=\frac{2(p-b)\cdot 2(p-c)}{4bc}$

$$\Rightarrow\sin^2\frac{A}{2}=\frac{(p-b)(p-c)}{bc}\Rightarrow\sin\frac{A}{2}=\pm\sqrt{\frac{(p-b)(p-c)}{bc}}$$

E_7) $A\in(0;\pi)$ pt că A unghi al $\triangle ABC$

$\Rightarrow\frac{A}{2}\in(0;\frac{\pi}{2})\Rightarrow\frac{A}{2}\in C_1 I\Rightarrow\sin\frac{A}{2}>0$

$$\Rightarrow\sin\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}$$

Prof. Ovidiu Bădescu