IDENTITĂȚI TRIGONOMETRICE FOLOSIND TRANSFORMAREA PRODUSELOR ÎN SUME ȘI A SUMELOR ÎN PRODUS

Arătați că:

1) $\frac{\cos 5\alpha - \cos\alpha}{\cos 3\alpha - \cos\alpha} = 4\cos^2\alpha - 1$

2) $\frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \cos 7\alpha} = \operatorname{tg} 4\alpha$

3) $\sin\alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha = 4\cos\alpha\cos 2\alpha\sin 4\alpha$

4) $\frac{\cos\alpha - \cos 3\alpha + \cos 5\alpha - \cos 7\alpha}{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha} = \operatorname{tg}\alpha$

5) $\sin\alpha + \sin 2\alpha + \ldots + \sin 10\alpha = \frac{\sin 5\alpha \sin\frac{11\alpha}{2}}{\sin\frac{\alpha}{2}}$

6) $\cos\alpha + \cos 2\alpha + \ldots + \cos 10\alpha = \frac{\sin\frac{11\alpha}{2}\cos 6\alpha}{\sin\frac{\alpha}{2}}$

7) $1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = 0$

8) $\frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha}{\cos\alpha - \cos 3\alpha + \cos 5\alpha - \cos 7\alpha} = \operatorname{ctg}\alpha$

Prof. Ovidiu Bădescu

Arătați că: $\frac{\cos 5\alpha - \cos\alpha}{\cos 3\alpha - \cos\alpha} = 4\cos^2\alpha - 1$

E_1) Folosim: $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

E_2) $\frac{\cos 5\alpha - \cos\alpha}{\cos 3\alpha - \cos\alpha} = \frac{-2\sin\frac{5\alpha+\alpha}{2}\sin\frac{5\alpha-\alpha}{2}}{-2\sin\frac{3\alpha+\alpha}{2}\sin\frac{3\alpha-\alpha}{2}} =$

$= \frac{\sin 3\alpha \sin 2\alpha}{\sin 2\alpha \sin\alpha} = \frac{\sin 3\alpha}{\sin\alpha}$

E_3) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ / se deduce din $\sin(2\alpha+\alpha) = \dots$)

E_4) $\frac{\sin 3\alpha}{\sin\alpha} = \frac{3\sin\alpha - 4\sin^3\alpha}{\sin\alpha} = \frac{\sin\alpha(3-4\sin^2\alpha)}{\sin\alpha}$

$= 3 - 4\sin^2\alpha =$

$= 3 - 4(1-\cos^2\alpha) = 3 - 4 + 4\cos^2\alpha =$

$= 4\cos^2\alpha - 1$

Prof. Ovidiu Bădescu

Arătați că

$$\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}{\cos\alpha-\cos 3\alpha+\cos 5\alpha-\cos 7\alpha}=\operatorname{ctg}\alpha$$

E1) $E=\dfrac{(\sin\alpha+\sin 3\alpha)+(\sin 5\alpha+\sin 7\alpha)}{(\cos\alpha-\cos 3\alpha)+(\cos 5\alpha-\cos 7\alpha)}$

$$E=\frac{2\sin\frac{\alpha+3\alpha}{2}\cos\frac{\alpha-3\alpha}{2}+2\sin\frac{5\alpha+7\alpha}{2}\cos\frac{5\alpha-7\alpha}{2}}{-2\sin\frac{\alpha+3\alpha}{2}\sin\frac{\alpha-3\alpha}{2}-2\sin\frac{5\alpha+7\alpha}{2}\sin\frac{5\alpha-7\alpha}{2}}$$

$$E=\frac{2\cos\alpha\,(\sin(2\alpha)+\sin(6\alpha))}{-2\sin(-\alpha)\,(\sin(2\alpha)+\sin(6\alpha))}$$

$$E=\frac{2\cos\alpha}{2\sin\alpha}=\operatorname{ctg}\alpha$$

Prof. Ovidiu Badescu

Arătați că:

$$\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha=4\cos\alpha\cos 2\alpha\sin 4\alpha$$

E_1) Folosim $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

E_2) $M_s=(\sin\alpha+\sin 7\alpha)+(\sin 3\alpha+\sin 5\alpha)=$

$$=2\sin\frac{\alpha+7\alpha}{2}\cos\frac{\alpha-7\alpha}{2}+2\sin\frac{3\alpha+5\alpha}{2}\cos\frac{3\alpha-5\alpha}{2}$$

$$=2\sin 4\alpha\cos(-3\alpha)+2\sin 4\alpha\cos(-\alpha)$$

E_3) dar, $\cos(-x)=\cos x$

$$\Rightarrow M_s=2\sin 4\alpha(\cos 3\alpha+\cos\alpha)$$

E_4) Folosim $\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$$\Rightarrow M_s=2\sin 4\alpha\cdot 2\cos\frac{3\alpha+\alpha}{2}\cos\frac{3\alpha-\alpha}{2}$$

$$M_s=4\sin 4\alpha\cos 2\alpha\cos\alpha$$

Prof. Ovidiu Bădescu

Arătați că:

$$\frac{\cos\alpha-\cos 3\alpha+\cos 5\alpha-\cos 7\alpha}{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}=\operatorname{tg}\alpha$$

E.) $M_S=\dfrac{\cos\alpha-\cos 3\alpha+\cos 5\alpha-\cos 7\alpha}{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}=$

$$=\frac{-2\sin\frac{\alpha+7\alpha}{2}\sin\frac{\alpha-7\alpha}{2}+\left(-2\sin\frac{5\alpha+3\alpha}{2}\sin\frac{5\alpha-3\alpha}{2}\right)}{2\sin\frac{\alpha+7\alpha}{2}\cos\frac{\alpha-7\alpha}{2}+2\sin\frac{3\alpha+5\alpha}{2}\cos\frac{3\alpha-5\alpha}{2}}$$

$$=\frac{-2\sin 4\alpha\sin(-3\alpha)-2\sin 4\alpha\sin\alpha}{2\sin 4\alpha\cos(-3\alpha)+2\sin 4\alpha\cos(-\alpha)}=$$

$$=\frac{2\sin 4\alpha\sin 3\alpha-2\sin 4\alpha\sin\alpha}{2\sin 4\alpha\cos 3\alpha+2\sin 4\alpha\cos\alpha}=$$

$$=\frac{2\sin 4\alpha(\sin 3\alpha-\sin\alpha)}{2\sin 4\alpha(\cos 3\alpha+\cos\alpha)}=$$

$$=\frac{2\sin\frac{3\alpha-\alpha}{2}\cos\frac{3\alpha+\alpha}{2}}{2\cos\frac{3\alpha+\alpha}{2}\cos\frac{3\alpha-\alpha}{2}}=\frac{\sin\alpha\cos 2\alpha}{\cos 2\alpha\cos\alpha}=\operatorname{tg}\alpha$$

Prof. Ovidiu Badescu

Arătați că:

$$\sin\alpha+\sin 2\alpha+..+\sin 10\alpha=\frac{\sin 5\alpha\sin\frac{11\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$E_1)\ M_S=\sin\alpha+\sin 2\alpha+..+\sin 10\alpha$

$E_2)$ ca și metodă generală, se înmulțește M_S cu $\sin\frac{2\alpha-\alpha}{2}$ adică cu $\sin\frac{\alpha}{2}$ și transf produsele în sume folosind $\sin a\sin b=\frac{-1}{2}(\cos(a+b)-\cos(a-b))$

$E_3)\ M_S=\sin\alpha+\sin 2\alpha+...+\sin 10\alpha\ /\cdot\sin\frac{\alpha}{2}$

$$\Rightarrow M_S\cdot\sin\frac{\alpha}{2}=\sin\alpha\sin\frac{\alpha}{2}+\sin 2\alpha\sin\frac{\alpha}{2}+..+\sin 10\alpha\sin\frac{\alpha}{2}$$

$$\Rightarrow M_S\cdot\sin\frac{\alpha}{2}=\frac{-1}{2}\left[\cos\left(\alpha+\frac{\alpha}{2}\right)-\cos\left(\alpha-\frac{\alpha}{2}\right)+\cos\left(2\alpha+\frac{\alpha}{2}\right)-\cos\left(2\alpha-\frac{\alpha}{2}\right)+\right.$$
$$\left.\cdots+\cos\left(10\alpha+\frac{\alpha}{2}\right)-\cos\left(10\alpha-\frac{\alpha}{2}\right)\right]$$

$$M_S\cdot\sin\frac{\alpha}{2}=-\frac{1}{2}\left[\cos\frac{3\alpha}{2}-\cos\frac{\alpha}{2}+\cos\frac{5\alpha}{2}-\cos\frac{3\alpha}{2}+\cdots+\cos\frac{21\alpha}{2}-\cos\frac{19\alpha}{2}\right]$$

$$M_S\cdot\sin\frac{\alpha}{2}=-\frac{1}{2}\left[-\cos\frac{\alpha}{2}+\cos\frac{21\alpha}{2}\right]=-\frac{1}{2}\left[\cos\frac{21\alpha}{2}-\cos\frac{\alpha}{2}\right]$$

$E_4)$ dar, $\cos a-\cos b=-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$

$$\Rightarrow M_S\cdot\sin\frac{\alpha}{2}=-\frac{1}{2}\cdot(-2)\cdot\sin\frac{\frac{21\alpha}{2}+\frac{\alpha}{2}}{2}\sin\frac{\frac{21\alpha}{2}-\frac{\alpha}{2}}{2}$$

$$M_S\sin\frac{\alpha}{2}=\sin\frac{11\alpha}{2}\sin 5\alpha\Rightarrow M_S=\frac{\sin 5\alpha\sin\frac{11\alpha}{2}}{\sin\frac{\alpha}{2}}$$

Prof. Ovidiu Bădescu

Arătați că:

$$\cos\alpha+\cos 2\alpha+\ldots+\cos 11\alpha=\frac{\sin\frac{11\alpha}{2}\cos 6\alpha}{\sin\frac{\alpha}{2}}$$

—

E_1) Fie $S=\cos\alpha+\cos 2\alpha+\ldots+\cos 11\alpha$.

Ca și metodă de reținut, se înmulțește suma cu $\sin\frac{\alpha}{2}$, apoi transformăm fiecare produs în sumă folosind $\sin a\cos b=\frac{1}{2}(\sin(a+b)+\sin(a-b))$

E_2) $S\cdot\sin\frac{\alpha}{2}=\sin\frac{\alpha}{2}\cos\alpha+\sin\frac{\alpha}{2}\cos 2\alpha+\ldots+\sin\frac{\alpha}{2}\cos 11\alpha$

$$\Rightarrow S\sin\frac{\alpha}{2}=\frac{1}{2}\left[\sin\left(\frac{\alpha}{2}+\alpha\right)+\sin\left(\frac{\alpha}{2}-\alpha\right)\right]+\frac{1}{2}\left[\sin\left(\frac{\alpha}{2}+2\alpha\right)+\sin\left(\frac{\alpha}{2}-2\alpha\right)\right]+$$

$$+\ldots+\frac{1}{2}\left[\sin\left(\frac{\alpha}{2}+11\alpha\right)+\sin\left(\frac{\alpha}{2}-11\alpha\right)\right]$$

$$S\sin\frac{\alpha}{2}=\frac{1}{2}\left[\sin\frac{3\alpha}{2}+\sin\left(-\frac{\alpha}{2}\right)\right]+\frac{1}{2}\left[\sin\frac{5\alpha}{2}+\sin\left(-\frac{3\alpha}{2}\right)\right]+\ldots$$

$$+\frac{1}{2}\left[\sin\frac{23\alpha}{2}+\sin\left(-\frac{21\alpha}{2}\right)\right]$$

E_3) Dar, $\sin(-x)=-\sin x\Rightarrow$

$$\Rightarrow S\cdot\sin\frac{\alpha}{2}=\frac{1}{2}\left(\sin\frac{3\alpha}{2}-\sin\frac{\alpha}{2}+\sin\frac{5\alpha}{2}-\sin\frac{3\alpha}{2}+\ldots+\sin\frac{23\alpha}{2}-\sin\frac{21\alpha}{2}\right)$$

$$\Rightarrow S\cdot\sin\frac{\alpha}{2}=\frac{1}{2}\left(-\sin\frac{\alpha}{2}+\sin\frac{23\alpha}{2}\right)=\frac{1}{2}\left(\sin\frac{23\alpha}{2}-\sin\frac{\alpha}{2}\right)$$

E_4) Dar, $\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2}\Rightarrow$

$$S\sin\frac{\alpha}{2}=\frac{1}{2}\cdot 2\sin\frac{\frac{23\alpha}{2}-\frac{\alpha}{2}}{2}\cos\frac{\frac{23\alpha}{2}+\frac{\alpha}{2}}{2}$$

$$S\sin\frac{\alpha}{2}=\sin\frac{22\alpha}{4}\cos\frac{24\alpha}{4}\Rightarrow S\sin\frac{\alpha}{2}=\sin\frac{11\alpha}{2}\cos 6\alpha$$

$$\Rightarrow S=\frac{\sin\frac{11\alpha}{2}\cos 6\alpha}{\sin\frac{\alpha}{2}}$$

Prof. Ovidiu Badescu

Arătați că:

$$1+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=0$$

E_1) Egalitatea este echivalentă cu

$$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=-1$$

E_2) Fie $N=\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}$, ideea este să înmulțim cu ceva astfel încât transformând produsele în sume să se reducă

E_3) $N\cdot\sin\frac{\pi}{5}=\sin\frac{\pi}{5}\cos\frac{2\pi}{5}+\sin\frac{\pi}{5}\cos\frac{4\pi}{5}+\dots+\sin\frac{\pi}{5}\cos\frac{8\pi}{5}$

E_4) Dar, $\sin a\cos b=\frac{1}{2}[\sin(a+b)+\sin(a-b)]$

$$\Rightarrow N\cdot\sin\frac{\pi}{5}=\frac{1}{2}\Big[\Big(\sin\frac{3\pi}{5}+\sin\Big(-\frac{\pi}{5}\Big)\Big)+\Big(\sin\frac{5\pi}{5}+\sin\Big(-\frac{3\pi}{5}\Big)\Big)+$$

$$+\Big(\sin\frac{7\pi}{5}+\sin\Big(-\frac{5\pi}{5}\Big)\Big)+\sin\Big(\frac{9\pi}{5}+\sin\Big(-\frac{7\pi}{5}\Big)\Big)\Big]$$

E_5) Dar, $\sin(-x)=-\sin x$ și după reduceri de termeni

$$\Rightarrow N\cdot\sin\frac{\pi}{5}=\frac{1}{2}\Big(\sin\frac{9\pi}{5}-\sin\frac{\pi}{5}\Big)$$

E_6) Dar, $\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$

$$\Rightarrow N\cdot\sin\frac{\pi}{5}=\frac{1}{2}\cdot 2\sin\frac{4\pi}{5}\cos\pi$$

E_7) Dar, $\sin\frac{4\pi}{5}=\sin\Big(\pi-\frac{\pi}{5}\Big)=\sin\pi\cos\frac{\pi}{5}-\cos\pi\sin\frac{\pi}{5}=\sin\frac{\pi}{5}$

$$\Rightarrow N\cdot\sin\frac{\pi}{5}=\sin\frac{\pi}{5}\cdot(-1)\Rightarrow N=-1$$

Prof. Ovidiu Bădescu

Arătați că:

$$\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}{\cos\alpha-\cos 3\alpha+\cos 5\alpha-\cos 7\alpha}=\operatorname{ctg}\alpha$$

$E_1)\ E=\dfrac{(\sin\alpha+\sin 7\alpha)+(\sin 3\alpha+\sin 5\alpha)}{(\cos\alpha-\cos 7\alpha)-(\cos 3\alpha-\cos 5\alpha)}$

$$E=\frac{2\sin\frac{\alpha+7\alpha}{2}\cos\frac{\alpha-7\alpha}{2}+2\sin\frac{3\alpha+5\alpha}{2}\cos\frac{3\alpha-5\alpha}{2}}{-2\sin\frac{\alpha+7\alpha}{2}\sin\frac{\alpha-7\alpha}{2}+2\sin\frac{3\alpha+5\alpha}{2}\sin\frac{3\alpha-5\alpha}{2}}$$

$$E_2)\ E=\frac{2\sin 4\alpha(\cos(3\alpha)+\cos\alpha)}{-2\sin 4\alpha(-\sin 3\alpha+\sin\alpha)}$$

$$E=\frac{2\cos\frac{3\alpha+\alpha}{2}\cos\frac{3\alpha-\alpha}{2}}{+(\sin 3\alpha-\sin\alpha)}$$

$$E=\frac{2\cos(2\alpha)\cos\alpha}{2\sin\frac{3\alpha-\alpha}{2}\cos\frac{3\alpha+\alpha}{2}}=\frac{\cos\alpha}{\sin\alpha}=\operatorname{ctg}\alpha$$

Prof. Ovidiu Badescu

Arătați că în $\triangle ABC$ are loc:

$$-a\cos(B-C) = b\cos B + c\cos C$$

E_1) Din T. sinusurilor $\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

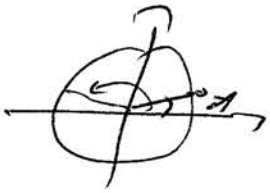
E_2) $T_S = 2R\sin A\cos(B-C)$

$$T_d = 2R\sin B\cos B + 2R\sin C\cos C =$$

$$= R(\sin 2B + \sin 2C) =$$

$$= R\cdot 2\cdot \sin\frac{2B+2C}{2}\cos\frac{2B-2C}{2} =$$

$$= 2R\sin(B+C)\cos(B-C)$$



E_3) $\sin(B+C) = \sin(\pi - A) \overset{\text{coresp}}{=\!=} \sin A$

$$\Rightarrow T_d = 2R\sin A\cos(B-C) \Rightarrow T_d = T_S$$

Prof. Ovidiu Bădescu

Arătați că în $\triangle ABC$ avem:

$$\cos A+\cos B+\cos C=1+\frac{r}{R}$$

E1) $T_S=\cos A+\cos B+\cos C=2\cos\frac{A-B}{2}\cos\frac{A+B}{2}+\cos C$

$A+B+C=\pi\Rightarrow A+B=\pi-C\Rightarrow\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$T_S=2\cos\frac{A-B}{2}\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)+\cos C=2\cos\frac{A-B}{2}\sin\frac{C}{2}+\cos C$

Dar, $\cos C=\cos\left(2\cdot\frac{C}{2}\right)=1-2\sin^2\frac{C}{2}\Rightarrow$

$T_S=2\cos\frac{A-B}{2}\sin\frac{C}{2}+1-2\sin^2\frac{C}{2}=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)$

$=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{\pi-(A+B)}{2}\right)=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$

$=1+2\sin\frac{C}{2}\cdot\left(-2\sin\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\sin\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)=$

$=1-4\sin\frac{C}{2}\sin\frac{A}{2}\sin\left(-\frac{B}{2}\right)=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

E2) Folosind formulele lui Heron $\Rightarrow$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}\cdot\sqrt{\frac{(p-a)(p-c)}{ac}}\cdot\sqrt{\frac{(p-a)(p-b)}{ab}}=$$

$$=\frac{\sqrt{p(p-a)(p-b)(p-c)}\cdot\sqrt{p(p-a)(p-b)(p-c)}}{p\cdot abc}=\frac{S^2}{p\cdot abc}=\frac{S}{p}\cdot\frac{S}{abc}$$

E3) $\left.\begin{matrix}S=\frac{abc}{4R}\\ r=\frac{S}{p}\end{matrix}\right\}\Rightarrow\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=r\cdot\frac{\frac{abc}{4R}}{abc}=\frac{r}{4R}$

E4) $T_S=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1+4\cdot\frac{r}{4R}=1+\frac{r}{R}$

Prof. Ovidiu Bădescu

Arătați că în ΔABC are loc

$$a\cos A + b\cos B + c\cos C = \frac{2pr}{R}$$

E_1) din T. sinus $\Rightarrow \frac{a}{\sin A} = 2R \Rightarrow \boxed{a = 2R\sin A}$, analog $\boxed{b = 2R\sin B}$, $\boxed{c = 2R\sin C}$

E_2) $T_S = 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C =$

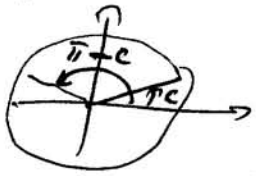
$= R(\sin 2A + \sin 2B + \sin 2C)$

E_3) $N_1 = \sin 2A + \sin 2B + \sin 2C = 2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2} + 2\sin C\cos C =$

$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

E_4) $A+B+C = \bar{\pi} \Rightarrow A+B = \bar{\pi} - C \Rightarrow \sin(A+B) = \sin(\bar{\pi} - C) = \sin C$

(din cerc sau desfac cu formula)



E_5) înlocuiesc în $N_1 \Rightarrow N_1 = 2\sin C\cos(A-B) + 2\sin C\cos C =$

$= 2\sin C(\cos(A-B) + \cos C) = 2\sin C \cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2}$

E_6) dar, $A+B+C = \bar{\pi} \Rightarrow A+C = \bar{\pi} - B$ și $B+C = \bar{\pi} - A$ și înlocuim

$\Rightarrow N_1 = 4\sin C\cdot\cos\frac{\bar{\pi}-2B}{2}\cos\frac{A-(\bar{\pi}-A)}{2} = 4\sin C\cos\left(\frac{\pi}{2} - B\right)\cos\left(A - \frac{\pi}{2}\right) =$

E_7) $\cos\left(\frac{\pi}{2} - B\right) = \sin B$ și $\cos\left(A - \frac{\pi}{2}\right) \xrightarrow{\text{cos pară}} \cos\left(\frac{\pi}{2} - A\right) = \sin A$

$\Rightarrow N_1 = 4\sin C\cdot\sin B\cdot\sin A = 4\sin A\sin B\sin C$

E_8) din T. sin $\Rightarrow \frac{a}{\sin A} = 2R \Rightarrow \sin A = \frac{a}{2R}$, analog $\sin B = \frac{b}{2R}$ etc

$\Rightarrow N_1 = 4\cdot\frac{abc}{8R^3}$, înlocuim în T_S

E_9) $T_S = R\cdot\frac{4abc}{8R^3} = \frac{abc}{2R^2}$

E_{10}) Dar, $S = \frac{abc}{4R}$ și $S = r\cdot p \Rightarrow \frac{abc}{4R} = r\cdot p$ $\Bigg\} \Rightarrow T_S = \frac{abc}{4R}\cdot\frac{2}{R} = \frac{2rp}{R}$

Prof. Ovidiu Bădescu

Arătați că dacă în ΔABC avem

$$\cos B + \cos C = \sin B + \sin C,$$

atunci ΔABC este dreptunghic

$E_1)$ $\cos B + \cos C = \sin B + \sin C \iff$

$$2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$$

$$\iff 2\cos\frac{B-C}{2}\left(\cos\frac{B+C}{2} - \sin\frac{B+C}{2}\right) = 0$$

$$\iff 2\cos\frac{B-C}{2}\left(\cos\frac{\pi-A}{2} - \sin\frac{\pi-A}{2}\right) = 0$$

$$\iff 2\cos\frac{B-C}{2}\left(\sin\frac{A}{2} - \sin\frac{\pi-A}{2}\right) = 0$$

$$\iff 2\cos\frac{B-C}{2}\, 2\sin\frac{\frac{A}{2}-\frac{\pi-A}{2}}{2}\cos\frac{\frac{A}{2}+\frac{\pi-A}{2}}{2} = 0$$

$$\iff 4\cos\frac{B-C}{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)\underbrace{\cos\frac{\pi}{4}}_{=\frac{\sqrt{2}}{2}} = 0$$

$E_2) \Rightarrow$

$\cos\frac{B-C}{2} = 0 \Rightarrow \frac{B-C}{2} = \frac{\pi}{2} \iff B-C = \pi \iff B = \pi + C$

$\Rightarrow B > 180^\circ$ Fals

sau

$\sin\left(\frac{A}{2}-\frac{\pi}{4}\right) = 0 \Rightarrow \frac{A}{2}-\frac{\pi}{4} \in \{0, \pi\} \Rightarrow \boxed{A = \frac{\pi}{2}}$ sau

$A = \frac{5\pi}{2}$ Fals

Prof. Ovidiu Bădescu

Dacă în ΔABC are loc

$$\frac{\sin B+\cos C}{\cos B+\sin C}=\operatorname{tg} B,$$ arătați că ΔABC este dreptunghic

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$E_1)$ $\frac{\sin B+\cos C}{\cos B+\sin C}=\frac{\sin B}{\cos B} \Leftrightarrow \cos B(\sin B+\cos C)=\sin B(\cos B+\sin C)$

$E_2)$ $\sin B\cos B+\cos B\cos C=\sin B\cos B+\sin B\sin C$

$\Leftrightarrow \cos B\cos C-\sin B\sin C=0$

$\Leftrightarrow \cos(B+C)=0$

$E_3)$ B, C unghiuri în $\Delta ABC \Rightarrow B+C=\frac{\pi}{2}$

Cum $A+B+C=\pi \Rightarrow C=\frac{\pi}{2} \Rightarrow \Delta ABC$ este dreptunghic

Prof. Ovidiu Bădescu

Dacă în ΔABC avem

$$1+\operatorname{ctg}\left(\frac{\pi}{4}-B\right)=\frac{2}{1-\operatorname{ctg}C}$$

arătați că ΔABC este dreptunghic

$E_1)$ $\operatorname{ctg}\left(\frac{\pi}{4}-B\right)=\frac{1}{\operatorname{tg}\left(\frac{\pi}{4}-B\right)}=\frac{1}{\frac{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg}B}{1+\operatorname{tg}\frac{\pi}{4}\operatorname{tg}B}}=\frac{1}{\frac{1-\operatorname{tg}B}{1+\operatorname{tg}B}}=$

$=\frac{1+\operatorname{tg}B}{1-\operatorname{tg}B}\Rightarrow T_1=1+\frac{1+\operatorname{tg}B}{1-\operatorname{tg}B}=\frac{1-\operatorname{tg}B+1+\operatorname{tg}B}{1-\operatorname{tg}B}=\frac{2}{1-\operatorname{tg}B}$

$E_2)$ Egalitatea devine:

$$\frac{2}{1-\operatorname{tg}B}=\frac{2}{1-\operatorname{ctg}C}\Leftrightarrow 1-\operatorname{tg}B=1-\operatorname{ctg}C\Leftrightarrow$$

$$\Leftrightarrow -\operatorname{tg}B=-\operatorname{ctg}C\Leftrightarrow \operatorname{tg}B=\operatorname{ctg}C\Leftrightarrow$$

$$\frac{\sin B}{\cos B}=\frac{\cos C}{\sin C}\Leftrightarrow$$

$$\sin B\sin C=\cos B\cos C\Leftrightarrow \cos B\cos C-\sin B\sin C=0$$

$$\Leftrightarrow \cos(B+C)=0$$

B, C unghiuri ale $\Delta ABC\Rightarrow B+C=\frac{\pi}{2}$

și cum $A+B+C=\pi\Rightarrow A=\frac{\pi}{2}$

$\Rightarrow \Delta ABC$ dreptunghic

Prof. Ovidiu Bădescu

Arătați că dacă în $\triangle ABC$ are loc

$$\sin A = \cos B + \cos C,$$

atunci $\triangle ABC$ este dreptunghic

$E_1)$ $\sin A = \cos B + \cos C \Leftrightarrow$

$$\sin A = 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} \Leftrightarrow 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\frac{\pi-A}{2}\cos\frac{B-C}{2}$$

$$\Leftrightarrow 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin\frac{A}{2}\cos\frac{B-C}{2}$$

$$\Leftrightarrow 2\sin\frac{A}{2}\left(\cos\frac{A}{2} - \cos\frac{B-C}{2}\right) = 0$$

$$\Leftrightarrow 2\sin\frac{A}{2}\cdot\left(-2\sin\frac{\frac{A}{2}+\frac{B-C}{2}}{2}\sin\frac{\frac{A}{2}-\frac{B-C}{2}}{2}\right) = 0$$

$$\Leftrightarrow -4\sin\frac{A}{2}\sin\frac{A+B-C}{4}\sin\frac{A-B+C}{4} = 0$$

$$\Leftrightarrow -4\sin\frac{A}{2}\sin\frac{\pi-2C}{4}\sin\frac{\pi-2B}{4} = 0$$

$E_2) \Rightarrow$

$\sin\frac{A}{2} = 0 \Leftrightarrow \frac{A}{2} \in \{0;\pi\} \Leftrightarrow A \in \{0;2\pi\}$ Fals

sau

$\sin\frac{\pi-2C}{4} = 0 \Leftrightarrow \frac{\pi-2C}{4} = 0 \Leftrightarrow C = \frac{\pi}{2}$

sau

$\sin\frac{\pi-2B}{4} = 0 \Leftrightarrow \frac{\pi-2B}{4} = 0 \Leftrightarrow B = \frac{\pi}{2}$

$\Rightarrow \triangle ABC$ dreptunghic

Prof. Ovidiu Badescu

Arătați că dacă în ΔABC avem

$$\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a}{bc}, \text{ atunci}$$

ΔABC este dreptunghic

E_1) din $a^2=b^2+c^2-2bc\cos A \Rightarrow \cos A=\frac{b^2+c^2-a^2}{2bc}$

$$\Rightarrow \frac{\cos A}{a}=\frac{b^2+c^2-a^2}{2abc}$$

E_2) Analog $\frac{\cos B}{b}=\frac{a^2+c^2-b^2}{2abc}$ și $\frac{\cos C}{c}=\frac{a^2+b^2-c^2}{2abc}$ și înlocuind $\Rightarrow$

$$T_1=\frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$$

E_3) Egalitatea din ipoteză devine

$$\frac{a^2+b^2+c^2}{2abc}=\frac{a}{bc} \Leftrightarrow a^2+b^2+c^2=2a^2$$

$$\Leftrightarrow b^2+c^2=a^2 \xrightarrow[\text{T. Pitag}]{\text{reciproca}} m(\hat{A})=\frac{\pi}{2}$$

$\Rightarrow \Delta ABC$ dreptunghic

Prof. Ovidiu Badescu

Arătați că dacă în $\triangle ABC$ avem

$$b^2 - c^2 = 2a^2, \text{ atunci } A \leq \frac{\pi}{6}$$

E_1) din T. cos $\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$ și înlocuim

$\Rightarrow b^2 - c^2 = 2(b^2 + c^2 - 2bc\cos A) \Leftrightarrow$

$b^2 - c^2 = 2b^2 + 2c^2 - 4bc\cos A \Leftrightarrow$

$4bc\cos A = b^2 + 3c^2 \Rightarrow$

$$\cos A = \frac{b^2 + 3c^2}{4bc}$$

E_2) din ineg mediilor $\Rightarrow m_g \leq m_a \Rightarrow \sqrt{xy} \leq \frac{x+y}{2}$

$\Rightarrow x + y \geq 2\sqrt{xy} \Rightarrow b^2 + 3c^2 \geq 2\sqrt{b^2 \cdot 3c^2}$

$\Rightarrow b^2 + 3c^2 \geq 2bc\sqrt{3} \;/:4bc$

$$\frac{b^2 + 3c^2}{4bc} \geq \frac{\sqrt{3}}{2} \Rightarrow \cos A \geq \frac{\sqrt{3}}{2} \Rightarrow \cos A \geq \cos\frac{\pi}{6}$$

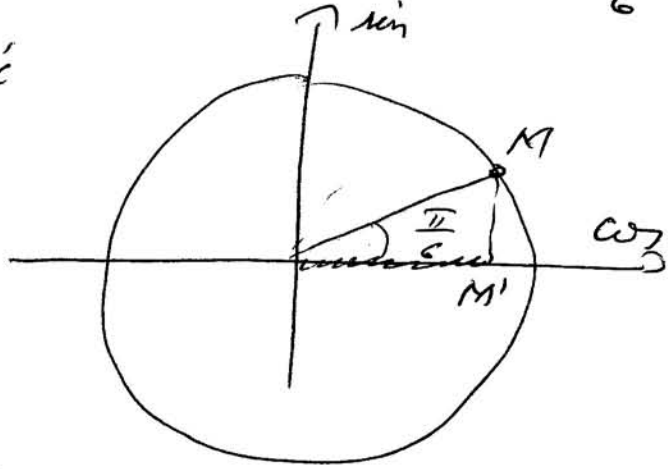
E_3) Din cercul trigonometric

$\Rightarrow \cos A \geq \cos\frac{\pi}{6}$ ar

trebui ca $A \in (0; \frac{\pi}{6})$

$\Rightarrow A \leq \frac{\pi}{6}$

(A că atunci OM' se mărește)



Prof. Ovidiu Badescu

Stabiliți natura ΔABC dacă

$$\frac{a+c}{b} = \operatorname{ctg}\frac{B}{2}$$

E1) $T_S = \frac{a}{b} + \frac{c}{b}$

Din T. sinusurilor $\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\Rightarrow \frac{a}{b} = \frac{\sin A}{\sin B}$ și $\frac{c}{b} = \frac{\sin C}{\sin B}$

$\Rightarrow T_S = \frac{\sin A + \sin C}{\sin B}$

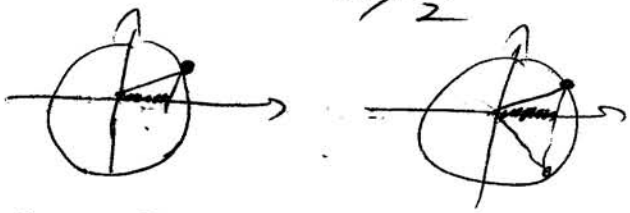
E2) Egalitatea din ipoteză devine

$$\frac{\sin A + \sin C}{\sin B} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} \Leftrightarrow \frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{2\sin\frac{B}{2}\cos\frac{B}{2}} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}$$

$$\Leftrightarrow \frac{\sin\frac{\pi-B}{2}\cos\frac{A-C}{2}}{\cos\frac{B}{2}} = \cos\frac{B}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\cos\frac{A-C}{2}}{\cos\frac{B}{2}} = \cos\frac{B}{2} \Leftrightarrow \frac{\cos\frac{B}{2}\cos\frac{A-C}{2}}{\cos\frac{B}{2}} = \cos\frac{B}{2}$$

$\Leftrightarrow \cos\frac{A-C}{2} = \cos\frac{B}{2}$.



E3) din cerc trig $\Rightarrow \frac{A-C}{2} = \frac{B}{2}$ sau $\frac{A-C}{2} = -\frac{B}{2}$

$\Leftrightarrow (A-C=B)$ sau $(A-C=-B) \Leftrightarrow (A=B+C)$ sau $A+B=C$

$\Rightarrow \left(A=\pi-A \Rightarrow A=\frac{\pi}{2}\right)$ sau $\left(\pi-C=C \Rightarrow C=\frac{\pi}{2}\right)$

$\Rightarrow \Delta ABC$ dreptunghic

Prof. Ovidiu Bădescu

Arătați că dacă ΔABC în care $\exists$ relația

$$\sin B+\cos B=\sin C+\cos C$$

$\Rightarrow \Delta ABC$ dreptunghic sau isoscel

$E_1)\ \sin B+\cos B=\sin C+\cos C \mid ()^2$

$\Rightarrow \sin^2 B+2\sin B\cos B+\cos^2 B=\sin^2 C+2\sin C\cos C+\cos^2 C$

$$1+\sin 2B=1+\sin 2C$$

$E_2)\ \sin 2B=\sin 2C \Rightarrow$

$2B=2C \Rightarrow B=C$ Δ isoscel

sau

$2B=\pi-2C$

$\Rightarrow 2B+2C=\pi \mid :2 \Rightarrow B+C=\frac{\pi}{2}$

$\Rightarrow A=90^\circ \Rightarrow$ dreptt.

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mod II : $E_1)\ \sin B-\sin C+\cos B-\cos C=0$

$E_2)\ 2\sin\frac{B-C}{2}\cos\frac{B+C}{2}-2\sin\frac{B+C}{2}\sin\frac{B-C}{2}=0$

$E_2)\ 2\sin\frac{B-C}{2}\cdot\left(\cos\frac{B+C}{2}-\sin\frac{B+C}{2}\right)=0$

$E_3)$ Cob I : $\sin\frac{B-C}{2}=0 \Rightarrow B-C=0 \Rightarrow B=C$ Δ isoscel

$E_4)$ Cob II : $\cos\frac{B+C}{2}=\sin\frac{B+C}{2} \Rightarrow \frac{B+C}{2}=\frac{\pi}{4}$

$\Rightarrow B+C=\frac{\pi}{2} \Rightarrow A=90^\circ \Rightarrow \Delta$ dreptt.

Prof. Ovidiu Băgoșcu

Stabiliți natura ΔABC dacă

$$a = 2b\cos C$$

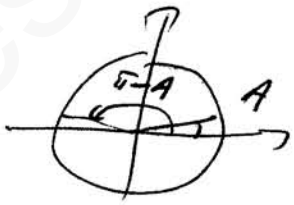
E_1) Din T. sin $\Rightarrow a = 2R\sin A,\ b = 2R\sin B$

$\Rightarrow 2R\sin A = 4R\sin B\cos C$

$\Leftrightarrow \sin A = 2\sin B\cos C$

E_2) $\sin A = 2\cdot\frac{1}{2}\left[\sin(B+C)+\sin(B-C)\right]$

$\sin A = \sin(\pi - A) + \sin(B-C)$



E_3) Din cerc, $\sin(\pi - A) = \sin A$

$\Rightarrow \sin A = \sin A + \sin(B-C)$

$\Leftrightarrow \sin(B-C) = 0$

$B, C \in (0, \pi) \Rightarrow B - C \in (-\pi, \pi)$ $\Big\}\Rightarrow B - C = 0 \Rightarrow B = C$

$\Rightarrow \Delta ABC$ isoscel

Prof. Ovidiu Badescu

Stabiliți natura ΔABC dacă

$$a = 2b \sin\frac{A}{2},\ A > \frac{\pi}{3}$$

E_1) din T. sin $\Rightarrow a = 2R\sin A,\ b = 2R\sin B$

E_2) $2R\sin A = 2\cdot 2R\sin B\sin\frac{A}{2} \ \big|:2R$

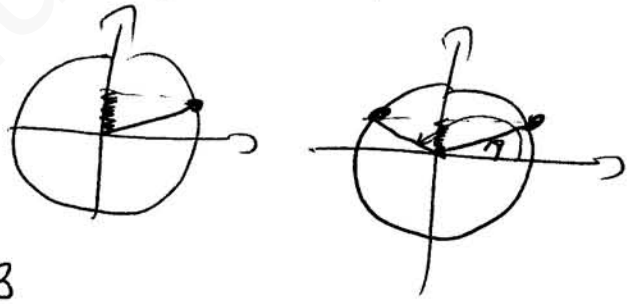
$\Rightarrow \sin A = 2\sin B\sin\frac{A}{2}$

E_3) Dar, $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ $\Big\}\Rightarrow$

$2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin B\sin\frac{A}{2} \ \big|:2\sin\frac{A}{2}$

$\Rightarrow \cos\frac{A}{2} = \sin B \Rightarrow \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin B$

$\Rightarrow \sin\frac{\pi - A}{2} = \sin B$



E_4) din arc $\Rightarrow$

$\frac{\pi - A}{2} = B$ sau $\frac{\pi - A}{2} = \pi - B$

$\Leftrightarrow (\pi = 2B + A)$ sau $\pi - A = 2\pi - B \Leftrightarrow$

$\Leftrightarrow (A + B + C = 2B + A)$ sau $(B - A = \pi) \Leftrightarrow$

$\Leftrightarrow (C = B)$ sau $(B - A = A + B + C)$

(Δ isoscel) sau ($2A + C = 0$ fals)

$\Rightarrow \Delta ABC$ isoscel

Prof. Ovidiu Bădescu

Stabiliți natura ΔABC dacă

$$a^2+b^2+c^2=8R^2$$

E_1) din T. sinusurilor, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

$$\Rightarrow \begin{cases} a^2=4R^2\sin^2 A \\ b^2=4R^2\sin^2 B \\ c^2=4R^2\sin^2 C \end{cases}$$ și înlocuind

E_2) $4R^2(\sin^2 A+\sin^2 B+\sin^2 C)=8R^2$

$$\Leftrightarrow \sin^2 A+\sin^2 B+\sin^2 C=2$$

E_3) Dar, $\sin^2 x=\frac{1-\cos 2x}{2}\Rightarrow$

$$\Rightarrow \frac{1-\cos 2A}{2}+\frac{1-\cos 2B}{2}+\frac{1-\cos 2C}{2}=2 \mid \cdot 2$$

$$3-(\cos 2A+\cos 2B+\cos 2C)=4 \Leftrightarrow 1+\cos 2A+\cos 2B+\cos 2C=0$$

E_4) $2\cos\frac{2A+2B}{2}\cos\frac{2A-2B}{2}+\cos 2C+1=0$

$$2\cos(A+B)\cos(A-B)+2\cos^2 C=0$$

$$2\cos(\pi-C)\cos(A-B)+2\cos^2 C=0 \mid :2$$

E_5) Dar cum, $\cos(\pi-C)=-\cos C \Rightarrow -\cos C\cos(A-B)+\cos^2 C=0$

$$\Rightarrow \cos C(\cos C-\cos(A-B))=0 \Rightarrow \begin{cases} \cos C=0 \Rightarrow C=\frac{\pi}{2} \\ \text{sau} \\ \cos C=\cos(A-B) \end{cases}$$

E_6) Din a (2)-a relație $\Rightarrow (C=A-B)$ sau $(C=-(A-B))$

$\Leftrightarrow (B+C=A)$ sau $(A+C=B) \Leftarrow \left(\pi-A=A\Rightarrow A=\frac{\pi}{2}\right)$ sau $\left(B=\frac{\pi}{2}\right)$

$\Rightarrow \Delta ABC$ dreptunghic

Prof. Ovidiu Bădescu

Studiați natura $\triangle ABC$ dacă

$$a \cdot c \cdot \sin B \cdot \operatorname{tg} B = b^2$$

E_1) Din T. sinusurilor, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$$\Rightarrow (2R \sin A)(2R \sin C) \sin B \cdot \frac{\sin B}{\cos B} = 4R^2 \sin^2 B \,/ : \sin^2 B$$

$$\Leftrightarrow \sin A \sin C = \cos B$$

E_2) $A + B + C = \bar{u} \Rightarrow B = \bar{u} - (A + C)$

$$\Rightarrow \sin A \sin C = \cos(\bar{u} - (A + C)) \xrightarrow{\text{cer}}$$

$$\sin A \sin C = -\cos(A + C)$$

$$\Rightarrow \sin A \sin C = -(\cos A \cos C - \sin A \sin C)$$

$$\sin A \sin C = -\cos A \cos C + \sin A \sin C$$

$$\cos A \cos C = 0 \Rightarrow \begin{cases} \cos A = 0 \Rightarrow A = \frac{\bar{u}}{2} \\ \text{sau} \qquad \text{sau} \\ \cos C = 0 \quad C = \frac{\bar{u}}{2} \end{cases}$$

$\Rightarrow \triangle ABC$ dreptunghic

Prof. Ovidiu Bădescu

Dacă $A+B+C=\pi$, arătați că

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

E1) $M_S=\sin A+\sin B+\sin C=2\sin\frac{A+C}{2}\cos\frac{A-C}{2}+\sin B$

E2) Dar, $A+B+C=\pi\Rightarrow\frac{A+C}{2}=\frac{\pi}{2}-\frac{B}{2}\Rightarrow\sin\frac{A+C}{2}=\cos\frac{B}{2}$

și $\sin B=\sin\left(2\cdot\frac{B}{2}\right)=2\sin\frac{B}{2}\cos\frac{B}{2}$

E3) $M_S=2\cos\frac{B}{2}\cos\frac{A-C}{2}+2\sin\frac{B}{2}\cos\frac{B}{2}=$

$=2\cos\frac{B}{2}\left[\cos\frac{A-C}{2}+\sin\frac{B}{2}\right]$

E4) $\sin\frac{B}{2}=\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)=\cos\frac{A+B+C-B}{2}=\cos\frac{A+C}{2}$

E5) $M_S=2\cos\frac{B}{2}\left[\cos\frac{A-C}{2}+\cos\frac{A+C}{2}\right]=$

$=2\cos\frac{B}{2}\cdot 2\cos\frac{\frac{A-C+A+C}{2}}{2}\cos\frac{\frac{A-C-A-C}{2}}{2}$

$=2\cos\frac{B}{2}\cdot 2\cos\frac{A}{2}\cos\left(-\frac{C}{2}\right)$

$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Obs.: S-a folosit: $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$,

$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$; $\sin x=\cos\left(\frac{\pi}{2}-x\right)$

și $\cos(-x)=\cos x$

Prof. Ovidiu Badescu

Dacă $A+B+C=\pi$, arătați că:

$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$E_1)\ M_S=\cos A+\cos B+\cos C=2\cos\frac{A+C}{2}\cos\frac{A-C}{2}+\cos B$

$=2\cos\frac{\pi-B}{2}\cos\frac{A-C}{2}+1-2\sin^2\frac{B}{2}$

$E_2)\ M_S=2\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)\cos\frac{A-C}{2}+1-2\sin^2\frac{B}{2}$

$M_S=1+2\sin\frac{B}{2}\cos\frac{A-C}{2}-2\sin^2\frac{B}{2}$

$E_3)\ M_S=1+2\sin\frac{B}{2}\left(\cos\frac{A-C}{2}-\sin\frac{B}{2}\right)$

$E_4)\ A+B+C=\pi\Rightarrow B=\pi-(A+C)\Rightarrow\frac{B}{2}=\frac{\pi}{2}-\frac{A+C}{2}$

$\Rightarrow\sin\frac{B}{2}=\sin\left(\frac{\pi}{2}-\frac{A+C}{2}\right)=\cos\frac{A+C}{2}$

$E_5)\ M_S=1+2\sin\frac{B}{2}\left[\cos\frac{A-C}{2}-\cos\frac{A+C}{2}\right]$

$M_S=1+2\sin\frac{B}{2}\cdot\left[-2\sin\frac{\frac{A-C+A+C}{2}}{2}\sin\frac{\frac{A-C-A-C}{2}}{2}\right]$

$M_S=1-4\sin\frac{B}{2}\sin\frac{A}{2}\sin\left(-\frac{C}{2}\right)$

$M_S=1+4\sin\frac{B}{2}\sin\frac{A}{2}\sin\frac{C}{2}$

Prof. Ovidiu Badescu

Dacă $A+B+C=\pi$, arătați că:

$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

E1) $M_S=\sin 2A+\sin 2B+\sin 2C=2\sin\frac{2A+2C}{2}\cos\frac{2A-2C}{2}+\sin 2B$

$=2\sin(A+C)\cos(A-C)+2\sin B\cos B$

E2) Dar, $A+B+C=\pi\Rightarrow A+C=\pi-B\Rightarrow$

$\Rightarrow\sin(A+C)=\sin(\pi-B)=\sin B$

E3) $M_S=2\sin B\cos(A-C)+2\sin B\cos B=$

$=2\sin B\left[\cos(A-C)+\cos B\right]=$

$=2\sin B\cdot 2\cdot\cos\frac{A-C+B}{2}\cos\frac{A-C-B}{2}$

E4) Dar, $A+B+C=\pi\Rightarrow A+B=\pi-C$

$B+C=\pi-A$

E5) $M_S=2\sin B\cdot 2\cos\frac{A+B-C}{2}\cos\frac{A-(B+C)}{2}$

$=4\sin B\cos\frac{\pi-2C}{2}\cos\frac{A-(\pi-A)}{2}=$

$=4\sin B\cos\left(\frac{\pi}{2}-C\right)\cos\left(A-\frac{\pi}{2}\right)$

$=4\sin B\sin C\cdot\cos\left(\frac{\pi}{2}-A\right)=4\sin A\sin B\sin C$

S-a folosit: $\cos(-x)=\cos x\Rightarrow\cos\left(A-\frac{\pi}{2}\right)=$

$=\cos\left(-\left(\frac{\pi}{2}-A\right)\right)=\cos\left(\frac{\pi}{2}-A\right)$

Prof. Ovidiu Băcescu

Arătați că $\sum \sin^2\frac{A}{2} = 1 - \frac{r}{2R}$

$E_1)$ $M_\Delta = \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = \frac{1-\cos A}{2} + \frac{1-\cos B}{2} + \frac{1-\cos C}{2} =$

$= \frac{3}{2} - \frac{1}{2}(\cos A + \cos B + \cos C)$

$E_2)$ $\cos A + \cos B + \cos C = 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} + 1 - 2\sin^2\frac{B}{2} =$

$= 1 + 2\cos\left(\frac{\pi}{2} - \frac{B}{2}\right)\cos\frac{A-C}{2} - 2\sin^2\frac{B}{2} = 1 + 2\sin\frac{B}{2}\left(\cos\frac{A-C}{2} - \sin\frac{B}{2}\right) =$

$= 1 + 2\sin\frac{B}{2}\left(\cos\frac{A-C}{2} - \sin\left(\frac{\pi}{2} - \frac{A+C}{2}\right)\right) = 1 + 2\sin\frac{B}{2}\left(\cos\frac{A-C}{2} - \cos\frac{A+C}{2}\right) =$

$= 1 + 2\sin\frac{B}{2}\cdot(-2)\sin\frac{A-C+A+C}{4}\sin\frac{A-C-A-C}{4} = 1 - 4\sin\frac{B}{2}\sin\frac{A}{2}\sin\left(-\frac{C}{2}\right)$

$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Rightarrow M_\Delta = \frac{3}{2} - \frac{1}{2}\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$\Rightarrow M_\Delta = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$E_3)$ Dar, $\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$ și analogele $\Rightarrow$

$M_\Delta = 1 - 2\cdot\sqrt{\frac{(p-a)^2(p-b)^2\cdot(p-c)^2\cdot p^2}{a^2b^2c^2\cdot p^2}} = 1 - 2\cdot\frac{S^2}{abc\cdot p}$ (1)

$E_4)$ $M_d = 1 - \frac{r}{2R} = 1 - \frac{\frac{S}{p}}{2\cdot\frac{abc}{4S}} = 1 - \frac{S}{p}\cdot\frac{2S}{abc} =$

$= 1 - \frac{2\cdot S^2}{p\,abc}$ (2)

$E_5)$ Din (1) și (2) $\Rightarrow$ egalitate

Prof. Ovidiu Bădescu

Arătați că

$$r_a + r_b + r_c = r + 4R$$

E1) $r_a + r_b + r_c - r = \underline{\frac{S}{p-a} + \frac{S}{p-b}} + \underline{\frac{S}{p-c} - \frac{S}{p}} =$

E2) $= \frac{S(p-b+p-a)}{(p-a)(p-b)} + \frac{S(p-p+c)}{p(p-c)} =$

$= \frac{S(2p-a-b)}{(p-a)(p-b)} + \frac{S \cdot c}{p(p-c)} =$

$= \frac{S(a+b+c-a-b)}{(p-a)(p-b)} + \frac{S \cdot c}{p(p-c)} =$

$= \frac{S \cdot c}{(p-a)(p-b)} \cdot \frac{p(p-c)}{p(p-c)} + \frac{S \cdot c}{p(p-c)} \cdot \frac{(p-a)(p-b)}{(p-a)(p-b)} =$

E3) $= \frac{Sc\left(p(p-c) + (p-a)(p-b)\right)}{p(p-a)(p-b)(p-c)} =$

$= \frac{S \cdot c \cdot \left(\frac{a+b+c}{2} \cdot \frac{a+b-c}{2} + \frac{-a+b+c}{2} \cdot \frac{a-b+c}{2}\right)}{S^2} =$

$= \frac{c \cdot \left(\frac{(a+b)^2 - c^2}{4} + \frac{c^2 - (a-b)^2}{4}\right)}{S} = \frac{c}{4S}\left(a^2 + 2ab + b^2 - c^2 + c^2 - a^2 + 2ab - b^2\right)$

$= \frac{c}{4S} \cdot 4ab = \frac{abc}{4S} \cdot 4 = 4R$

Prof. Ovidiu Bădescu

Dacă $A+B+C=\pi$, demonstrați că:

$$\sum \operatorname{tg} A = \prod \operatorname{tg} A$$

$E_1)$ $\sum \operatorname{tg} A = \operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C$

$\prod \operatorname{tg} A = \operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C$

$E_2)$ Dar, $A+B+C=\pi \Rightarrow A+B=\pi-C$

$E_3)$ $\operatorname{tg}(A+B)=\operatorname{tg}(\pi-C) \Rightarrow$

$$\frac{\operatorname{tg}A+\operatorname{tg}B}{1-\operatorname{tg}A\operatorname{tg}B} = \frac{\operatorname{tg}\pi-\operatorname{tg}C}{1-\operatorname{tg}\pi\operatorname{tg}C} \quad (*)$$

$E_4)$ Deoarece $\operatorname{tg}\pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0$ $(*)$

$$\frac{\operatorname{tg}A+\operatorname{tg}B}{1-\operatorname{tg}A\operatorname{tg}B} = \frac{0-\operatorname{tg}C}{1-0}$$

$$\Rightarrow \operatorname{tg}A+\operatorname{tg}B = (1-\operatorname{tg}A\operatorname{tg}B)\cdot(-\operatorname{tg}C)$$

$$\Rightarrow \operatorname{tg}A+\operatorname{tg}B = -\operatorname{tg}C+\operatorname{tg}A\operatorname{tg}B\operatorname{tg}C$$

$$\Rightarrow \operatorname{tg}A+\operatorname{tg}B+\operatorname{tg}C = \operatorname{tg}A\operatorname{tg}B\operatorname{tg}C$$

Prof. Ovidiu Badescu

Arătați că: $\sum \frac{a^2}{r_a} = 4(r+R)$

$E_1)\ M_A = \frac{a^2}{r_a} + \frac{b^2}{r_b} + \frac{c^2}{r_c} = \frac{a^2}{\frac{S}{p-a}} + \frac{b^2}{\frac{S}{p-b}} + \frac{c^2}{\frac{S}{p-c}} =$

$= \frac{a^2(p-a)+b^2(p-b)+c^2(p-c)}{S} = \frac{a^2\frac{-a+b+c}{2} + b^2\frac{a-b+c}{2} + c^2\frac{2ab-c}{2}}{S}$

$= \frac{-a^3+a^2b+a^2c+ab^2-b^3+b^2c+ac^2+bc^2-c^3}{2S} =$

$= \frac{-(a^3+b^3+c^3)+a^2b+a^2c+ab^2+ac^2+bc^2+b^2c}{2S} \quad (1)$

$E_2)\ M_d = 4(r+R) = 4\left(\frac{S}{p} + \frac{abc}{4S}\right) = 4\cdot\frac{4S^2+pabc}{4Sp} =$

$= \frac{4\cdot p(p-a)(p-b)(p-c)+p(abc)}{Sp} = \frac{4(p-a)(p-b)(p-c)+abc}{S} =$

$= \frac{4\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}+abc}{S} = \frac{(-a+b+c)(a-b+c)(a+b-c)+2abc}{2S}$

$= \frac{(-a^2+ab-ac+ab-b^2+bc+ac-bc+c^2)(a+b-c)+2abc}{2S} =$

$-a^3-a^2b+a^2c+a^2b+ab^2-abc-a^2c-abc+ac^2+a^2b+ab^2-abc-$
$-ab^2-b^3+b^2c+abc+b^2c-bc^2+a^2c+abc-ac^2-abc-b^2c+bc^2+$
$+ac^2+bc^2-c^3+2abc$

$= \frac{}{2S}$

$= \frac{-a^3-b^3-c^3+ab^2+ac^2+a^2b+a^2c+b^2c+bc^2}{2S} \quad (2)$

$E_3)$ Din (1) + (2) $\Rightarrow$ egalitate

Prof. Ovidiu Badescu

Dacă $a+b+c+d=2\pi$, să se arate

$$\cos a+\cos b+\cos c+\cos d=4\cos\frac{a+b}{2}\cos\frac{b+c}{2}\cos\frac{c+a}{2}$$

$E_1)\ M_1=(\cos a+\cos b)+(\cos c+\cos d)=$

$$=2\cos\frac{a+b}{2}\cos\frac{a-b}{2}+2\cos\frac{c+d}{2}\cos\frac{c-d}{2}$$

$E_2)$ trebuie să apară $\cos\frac{a+b}{2}$ la rezultat, deci

$a+b+c+d=2\pi \Rightarrow c+d=2\pi-(a+b) \Rightarrow \frac{c+d}{2}=\pi-\frac{a+b}{2}$

$$\Rightarrow \cos\frac{c+d}{2}=\cos\left(\pi-\frac{a+b}{2}\right)=-\cos\frac{a+b}{2}$$

$E_3)\ M_1=2\cos\frac{a+b}{2}\cos\frac{a-b}{2}-2\cos\frac{a+b}{2}\cos\frac{c-d}{2}=$

$$=2\cos\frac{a+b}{2}\left[\cos\frac{a-b}{2}-\cos\frac{c-d}{2}\right]$$

$E_4)\ M_1=2\cos\frac{a+b}{2}\left[-2\sin\frac{\frac{a-b}{2}+\frac{c-d}{2}}{2}\sin\frac{\frac{a-b}{2}-\frac{c-d}{2}}{2}\right]$

$$=2\cos\frac{a+b}{2}\cdot\left[-2\sin\frac{a+c-(b+d)}{4}\sin\frac{a+d-(b+c)}{4}\right]$$

$E_5)$ la rezultat trebuie să apară $b+c$ și $c+a$,

din $a+b+c+d=2\pi \Rightarrow b+d=2\pi-(a+c) \Rightarrow$

$$\Rightarrow \sin\frac{a+c-(b+d)}{4}=\sin\frac{a+c-(2\pi-(a+c))}{4}=\sin\left(\frac{a+c}{2}-\frac{\pi}{2}\right)=$$

$$=-\sin\left(\frac{\pi}{2}-\frac{a+c}{2}\right)=\cos\frac{a+c}{2}$$

din $a+b+c+d=2\pi \Rightarrow a+d=2\pi-(b+c) \Rightarrow \sin\frac{a+d-(b+c)}{4}=$

$$=\sin\frac{2\pi-2(b+c)}{4}=\sin\left(\frac{\pi}{2}-\frac{b+c}{2}\right)=\cos\frac{b+c}{2}$$

$a+b$ ... $a+c$ ... $\frac{b+c}{}$

Prof. Ovidiu Bădescu