# INEGALITĂȚI ELEMENTARE



Arătați că:

1) $a^2+b^2 \geq 2ab$, $a^2+b^2 \geq -2ab$

2) $a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$, $\forall a,b,c \geq 0$

3) $a^2+b^2+c^2 \geq ab+bc+ca$, $\forall a,b,c \in \mathbb{R}$

4) $(\sqrt{a}+\sqrt{b}+\sqrt{c})(a+b+c) \geq 9\sqrt{abc}$, $\forall a,b,c \geq 0$

5) $\sqrt[n]{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} < \frac{n+1}{2}$, $\forall n \in \mathbb{N}^*$, $n \geq 2$

6) $2\sqrt{a}+3\sqrt[3]{b} \geq 5\sqrt[5]{ab}$, $\forall a,b > 0$

7) $3\sqrt[3]{a}+4\sqrt[4]{b}+7\sqrt[7]{c} \geq 14\sqrt[14]{abc}$

8) $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, $\forall a,b,c \in \mathbb{R}$

9) $\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a} \leq 12$, $\forall a \in \left[\frac{3}{2}; \frac{50}{3}\right]$

10) $a^6+b^6+1 \geq 3a^2b^2$, $\forall a,b \in \mathbb{R}$

11) $ca+\frac{b}{c} \geq 2\sqrt{ab}$ și $\frac{a+bc^4}{2c^2} \geq \sqrt{ab}$

12) $ab(a+b)+bc(b+c)+ac(a+c) \geq 6abc$

13) $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \geq 6$, $\forall a,b,c > 0$

14) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ac}}+\frac{1}{\sqrt{ab}}$, $a,b,c > 0$

15) $\frac{ab}{c}+\frac{bc}{a}+\frac{ac}{b} \geq a+b+c$

Prof. Ovidiu Bădescu

Arătați că:

$$a+b+c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}, \forall a,b,c \geq 0$$

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$E_1$) $\frac{a+b}{2} \geq \sqrt{ab} \Rightarrow a+b \geq 2\sqrt{ab}$

Analog, $b+c \geq 2\sqrt{bc}$

și $c+a \geq 2\sqrt{ca}$

$E_2$| Adunând $\Rightarrow 2(a+b+c) \geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$

$\Rightarrow a+b+c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$

Prof. Ovidiu Badescu

Arătați că:

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(a+b+c) \geq 9\sqrt{abc}, \quad \forall a,b,c \geq 0$$

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$E_1)$ $\dfrac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3} \geq \sqrt[3]{\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c}} \Rightarrow$

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3} \geq \sqrt[6]{abc} \Rightarrow \boxed{\sqrt{a}+\sqrt{b}+\sqrt{c} \geq 3\sqrt[6]{abc}}$$

$E_2)$ $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow a+b+c \geq 3\sqrt[3]{abc}$

$E_3)$ înmulțind $\Rightarrow$

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(a+b+c) \geq 9\sqrt[6]{abc}\cdot\sqrt[3]{abc}$$

$E_4)$ dar, $\sqrt[6]{abc}\cdot\sqrt[3]{abc} = \sqrt[6]{abc}\cdot\sqrt[6]{(abc)^2} = \sqrt[6]{(abc)^3} =$

$$= \sqrt{abc} \Rightarrow$$

$E_5)$ $(\sqrt{a}+\sqrt{b}+\sqrt{c})(a+b+c) \geq 9\sqrt{abc}$

Prof. Ovidiu Badescu

Arătați că:

$$\sqrt[n]{1\cdot 2\cdot 3\cdot \ldots \cdot n} < \frac{n+1}{2},\ \forall n \in \mathbb{N}^*,\ n \geq 2$$

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$E_1$) din ineg mediilor $\Rightarrow$

$$\sqrt[n]{1\cdot 2\cdot 3\cdot \ldots \cdot n} \leq \frac{1+2+3+\ldots+n}{n}$$ cu egalitate

doar dacă toate numerele sunt egale

$E_2$) în cazul nostru, toate numerele nu sunt

fi egale $\Rightarrow \sqrt[n]{1\cdot 2\cdot \ldots \cdot n} < \frac{1+2+\ldots+n}{n}$

$E_3$) Dar, $\frac{1+2+\ldots+n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$

$E_4$) $\Rightarrow \sqrt[n]{1\cdot 2\cdot \ldots \cdot n} < \frac{n+1}{2},\ \forall n \geq 2$

Prof. Ovidiu Badescu

Arătați că:

$$2\sqrt{a}+3\sqrt[3]{b}\geq 5\sqrt[5]{ab}\ ,\ \forall a,b>0$$

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$E_1)\ M_1=2\sqrt{a}+3\sqrt[3]{b}=(\sqrt{a}+\sqrt{a})+(\sqrt[3]{b}+\sqrt[3]{b}+\sqrt[3]{b})$

$E_2$/ din ineg mediilor $\Rightarrow$

$$M_1\geq 5\sqrt[5]{(\sqrt{a}\cdot\sqrt{a})(\sqrt[3]{b}\cdot\sqrt[3]{b}\cdot\sqrt[3]{b})}$$

$$\Rightarrow M_1\geq 5\sqrt[5]{ab}$$

$$\Rightarrow 2\sqrt{a}+3\sqrt[3]{b}\geq 5\sqrt[5]{ab}$$

Prof. Ovidiu Badescu

Arătați că:

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2),\ \forall a,b,c \in \mathbb{R}$$

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E1) din ineg CBS

$$(a_1x_1+a_2x_2+a_3x_3)^2 \leq (a_1^2+a_2^2+a_3^2)(x_1^2+x_2^2+x_3^2) \Rightarrow$$

E2) $M_S = (1\cdot a+1\cdot b+1\cdot c)^2 \leq (1^2+1^2+1^2)(a^2+b^2+c^2)$

$$\Rightarrow (a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

Prof. Ovidiu Bădescu

Arătați că:

$$\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\leq 12,\ \forall a\in\left[\frac{3}{2},\frac{50}{3}\right]$$

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$E_1$) Ineg CBS

$$(a_1x_1+a_2x_2+\ldots+a_nx_n)^2\leq(a_1^2+\ldots+a_n^2)(x_1^2+\ldots+x_n^2)$$

$E_2$) în cazul nostru,

$$\left(1\cdot\sqrt{a+1}+1\cdot\sqrt{2a-3}+1\cdot\sqrt{50-3a}\right)^2\leq(1^2+1^2+1^2)(a+1+2a-3+50-3a)$$

$$\Leftrightarrow\left(\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\right)^2\leq 3\cdot 48$$

$$\Rightarrow\left(\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\right)^2\leq 144 \quad /\sqrt{\ldots}$$

$$\Rightarrow\left|\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\right|\leq 12$$

$E_3$) Dar, $\sqrt{a+1}$, $\sqrt{2a-3}$, $\sqrt{50-3a}$ pozitive

$$\Rightarrow\left|\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\right|=$$

$$=\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}$$

$$\Rightarrow\boxed{\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\leq 12}$$

Prof. Ovidiu Bădescu

Arătați că:

a) $ca+\frac{b}{c} \geq 2\sqrt{ab}$

b) $\frac{a+bc^4}{2c^2} \geq \sqrt{ab}$

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a) $E_1$) din ineg mediilor $\Rightarrow \frac{ca+\frac{b}{c}}{2} \geq \sqrt{ca \cdot \frac{b}{c}}$

$\Rightarrow ca+\frac{b}{c} \geq 2\sqrt{ab}$

b) $E_1$) folosim ineg. mediilor astfel:

$$\frac{a+bc^4}{2c^2} = \frac{a}{2c^2} + \frac{bc^4}{2c^2}$$

Dar, $\frac{a}{2c^2} + \frac{bc^4}{2c^2} \geq 2\sqrt{\frac{a}{2c^2} \cdot \frac{bc^4}{2c^2}}$

$\Rightarrow \frac{a}{2c^2} + \frac{bc^4}{2c^2} \geq 2\sqrt{\frac{ab}{4}} \Rightarrow$

$\Rightarrow \frac{a+bc^4}{2c^2} \geq \sqrt{ab}$

Prof Ovidiu Badescu

Arătați că:

$$ab(a+b)+bc(b+c)+ac(a+c) \geq 6abc$$

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$E_1$) Desfăcând parantezele $\Rightarrow$

$$M_S = a^2b+ab^2+b^2c+bc^2+a^2c+ac^2 \overset{\text{ineg mediilor}}{\geq}$$

$$\geq 6 \cdot \sqrt[6]{(a^2b)(ab^2)(b^2c)(bc^2)(a^2c)(ac^2)}$$

$$\Rightarrow M_S \geq 6 \cdot \sqrt[6]{a^6 \cdot b^6 \cdot c^6}$$

$$\Rightarrow M_S \geq 6abc$$

$E_2$) $\Rightarrow ab(a+b)+bc(b+c)+ac(a+c) \geq 6abc$

Prof. Ovidiu Bădescu

Arătați că:

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\geq 6,\ \forall a,b,c>0$$

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$E_1$) Folosim $x+\frac{1}{x}\geq 2\ \forall x>0$

$E_2$) $M_S=\left(\frac{a}{c}+\frac{b}{c}\right)+\left(\frac{b}{a}+\frac{c}{a}\right)+\left(\frac{c}{b}+\frac{a}{b}\right)$

$\Rightarrow M_S=\underbrace{\left(\frac{a}{c}+\frac{c}{a}\right)}_{\geq 2}+\underbrace{\left(\frac{b}{c}+\frac{c}{b}\right)}_{\geq 2}+\underbrace{\left(\frac{b}{a}+\frac{a}{b}\right)}_{\geq 2}$, deci

$E_3$) $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\geq 6,\ \forall a,b,c>0$

Prof. Ovidiu Bădescu

Arătați că:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ac}}+\frac{1}{\sqrt{ab}},$$

$\forall a,b,c>0$

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$E_1$) ineg mediilor $\Rightarrow$

$$\frac{1}{a}+\frac{1}{b}\geq 2\sqrt{\frac{1}{a}\cdot\frac{1}{b}}\Rightarrow\frac{1}{a}+\frac{1}{b}\geq\frac{2}{\sqrt{ab}}$$

Analog, $\frac{1}{b}+\frac{1}{c}\geq 2\sqrt{\frac{1}{b}\cdot\frac{1}{c}}\Rightarrow\frac{1}{b}+\frac{1}{c}\geq\frac{2}{\sqrt{bc}}$ și

$$\frac{1}{c}+\frac{1}{a}\geq 2\sqrt{\frac{1}{c}\cdot\frac{1}{a}}\Rightarrow\frac{1}{c}+\frac{1}{a}\geq\frac{2}{\sqrt{ac}}$$

$E_2$) adunăm $\Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 2\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ac}}\right)\Big|:2$

$$\Rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ac}}$$

Prof. Ovidiu Bădescu

Arătați că: $\frac{ab}{c}+\frac{bc}{a}+\frac{ac}{b}\geq a+b+c$

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$E_1$) din ineg mediilor $\Rightarrow$

$$\frac{ab}{c}+\frac{bc}{a}\geq 2\sqrt{\frac{ab}{c}\cdot\frac{bc}{a}}$$

$$\Rightarrow \frac{ab}{c}+\frac{bc}{a}\geq 2b$$

$E_2$) analog, $\frac{bc}{a}+\frac{ac}{b}\geq 2c$ și

$$\frac{ac}{b}+\frac{ab}{c}\geq 2a$$

$E_3$) adunând $\Rightarrow 2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ac}{b}\right)\geq 2(a+b+c)\,|:2$

$$\Rightarrow \frac{ab}{c}+\frac{bc}{a}+\frac{ac}{b}\geq a+b+c$$

Prof. Ovidiu Bădescu