INEGALITĂȚI DEMONSTRATE PRIN INDUCȚIE MATEMATICĂ

Demonstrați că:

1) $2^n \geq n+1$, $\forall n \in \mathbb{N}^*$

2) $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$, $\forall n \geq 2$

3) $\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{3n-2} \geq 1$, $\forall n \geq 1$

4) $\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$, $n \geq 1$

5) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$, $n \geq 1$

6) $2^n \geq n^2$, $\forall n \geq 5$

7) $3^n \geq 4n+1$, $\forall n \geq 2$

8) $\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$, $n \geq 1$

9) $\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n+1}{2n+2} > \frac{1}{2n+3}$, $n \geq 1$

10) $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$, $\forall n \geq 2$

11) $2^n \geq n^3$, $n \geq 10$

12) $\frac{3^n}{n+2} < \frac{(3n)!}{2(n!)^2}$, $n \geq 2$

13) $\frac{3}{5} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{4n-1}{4n+1} > \frac{1}{4n+3}$, $n \geq 1$

14) $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < \frac{4n}{2n+1}$ $n \geq 1$

15) $1^3 + 3^3 + \ldots + (2n-1)^3 < 2n^4$, $n \geq 1$

16) $\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} < \frac{2n-1}{n}$, $n \geq 4$

17) $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1$, $n \geq 1$

18) $3^n \geq n^3$, $\forall n \geq 4$

19) $4^n \geq n^3$, $\forall n \geq 3$

20) $2^n \geq 2n+1$, $\forall n \geq 4$

21) $\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-2} > \frac{1}{4}$, $\forall n \geq 3$

22) $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{4n} > 1$, $\forall n \geq 1$

23) $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{1}{5}$, $\forall n \geq 2$

Prof. Ovidiu Bădescu

Arătați că: $2^n \geq n+1, \forall n \in \mathbb{N}^*$

$E_1)$ $P(n): 2^n \geq n+1, \forall n \in \mathbb{N}^*$

$E_2)$ I $P(1): 2^1 \geq 1+1 \Leftrightarrow 2 \geq 2$ adev.

II Fie $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

$P(k): 2^k \geq k+1$

$P(k+1): 2^{k+1} \geq (k+1)+1$

$P(k+1): 2^{k+1} \geq k+2$

$E_3)$ Știm că $P(k)$ adev. $\Rightarrow 2^k \geq k+1 \mid \cdot 2$

$\Rightarrow 2^{k+1} \geq 2k+2$

E suficient să arăt că $2k+2 \geq k+2$

$\Leftrightarrow k \geq 0$ adev.

$\Rightarrow P(k+1)$ adev $\Rightarrow P(n)$ adev., $\forall n \geq 1$

Prof. Ovidiu Badescu

Demonstrați că:

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}>\frac{13}{24},\ \forall n\geq 2$$

$E_1)$ $P(2):\ \frac{1}{3}+\frac{1}{4}>\frac{13}{24}\ |\cdot 24 \Rightarrow 8+6>13$ adev.

$E_2)$ Fie $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

$P(k):\ \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{13}{24}$

$P(k+1):\ \frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k+2}>\frac{13}{24}$

$E_3)$ Știm că $P(k)$ adev. $\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{13}{24}$

Vrem să obținem $P(k+1)\Rightarrow$

$$\frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{13}{24}\ \Big|-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$\Rightarrow \frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k+2}>\frac{13}{24}-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$E_4)$ Suficient să arăt că: $\frac{13}{24}-\frac{2}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}>\frac{13}{24}$

$$\Rightarrow -\frac{2^{k+1}}{2k+2}+\frac{2^{k+2}}{2k+1}>0 \Rightarrow \frac{-(2k+1)+2k+2}{(k+2)(2k+1)}>0\Rightarrow$$

$$\Rightarrow \frac{-2k-1+2k+2}{(2k+2)(2k+1)}>0 \Leftrightarrow \frac{1}{(2k+2)(2k+1)}>0$$ adev.

$\Rightarrow P(k+1)$ adev. $\Rightarrow P(n)$ adev., $\forall n\geq 1$

Arătați că:

$$\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{3n-2}\geq 1,\ \forall n\geq 1$$

E_1) I P(1): $\frac{1}{1}\geq 1$ adev.

E_2) II Pp. P(k) adev $\overset{(?)}{\Rightarrow}$ P(k+1) adev unde

$P(k)$: $\frac{1}{k}+\frac{1}{k+1}+\dots+\frac{1}{3k-2}\geq 1$

$P(k+1)$: $\frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{3k+1}\geq 1$

E_3) Știm că P(k) adev $\Rightarrow$

$$\frac{1}{k}+\frac{1}{k+1}+\dots+\frac{1}{3k-2}\geq 1 \Big| -\frac{1}{k}+\frac{1}{3k-1}+\frac{1}{3k}+\frac{1}{3k+1}$$

$$\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{3k+1}\geq 1-\frac{1}{k}+\frac{1}{3k-1}+\frac{1}{3k}+\frac{1}{3k+1}$$

E_4) E suficient să arăt că

$$\frac{3}{\frac{1}{k}+\frac{1}{3k-1}+\frac{1}{3k}}+\frac{1}{3k+1}\geq 1$$

$$\Leftrightarrow -\frac{2}{3k}+\frac{1}{3k-1}+\frac{1}{3k+1}\geq 0 \Big| \cdot 3k(3k-1)(3k+1)$$

$$\Leftrightarrow -2(3k-1)(3k+1)+3k(3k+1)+3k(3k-1)\geq 0$$

$$\Leftrightarrow -2(9k^2-1)+9k^2+3k+9k^2-3k\geq 0$$

$$\Leftrightarrow -18k^2+2+18k^2\geq 0 \Leftrightarrow 2\geq 0$$ adev.

$\Rightarrow P(k+1)$ adev $\Rightarrow P(n)$ adev., $\forall n\geq 1$

Prof. Ovidiu Badescu

Arătați că: $\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}, \forall n \geq 2$

E_1) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$

E_2) I $P(2)$: $1 + \frac{1}{\sqrt{2}} > \sqrt{2} \mid \cdot \sqrt{2} \Rightarrow \sqrt{2} + 1 > 2$

$\sqrt{2} > 1$ adev.

II β_n. $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

$P(k)$: $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k}$

$P(k+1)$: $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

Știm că $P(k)$ adev. $\Rightarrow 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k} \mid + \frac{1}{\sqrt{k+1}}$

$\Rightarrow 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$

E suficient să arătăm că:

$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \mid \cdot \sqrt{k+1}$

$\Rightarrow \sqrt{k^2+k} + 1 > k+1 \mid ()^2$

$k^2 + k > k^2$ adev. $\Rightarrow P(k+1)$ adev.

$\Rightarrow P(n)$ adev., $\forall n \geq 2$

Prof. Ovidiu Bădescu

Arătați că: $1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}<2\sqrt{n}$, $\forall n\geq 2$

$E_1)$ $P(2)$: $1+\frac{1}{\sqrt{2}}<2\sqrt{2} \mid \cdot\sqrt{2} \Rightarrow \sqrt{2}+1<4$

$\sqrt{2}<3$ adev.

$E_2)$ P_k: $P(k)$ adev. $\overset{(?)}{\Rightarrow}$ $P(k+1)$ adev.

$P(k)$: $1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}}<2\sqrt{k}$

$P(k+1)$: $1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k+1}}<2\sqrt{k+1}$

Știm că $P(k)$ adev. $\Rightarrow$

$1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}}<2\sqrt{k} \mid +\frac{1}{\sqrt{k+1}}$

$\Rightarrow 1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k+1}}<2\sqrt{k}+\frac{1}{\sqrt{k+1}}$

Este suficient să arătăm că:

$2\sqrt{k}+\frac{1}{\sqrt{k+1}}<2\sqrt{k+1} \mid \cdot\sqrt{k+1}$

$2\sqrt{k^2+k}+1<2k+2$

$2\sqrt{k^2+k}<2k+1 \mid (\,)^2 \Rightarrow 4(k^2+k)<4k^2+4k+1$

$\Rightarrow 4k^2+4k<4k^2+4k+1 \Rightarrow 0<1$ adev.

Din ultimele $\Rightarrow P(n)$ adev.

Prof. Ovidiu Bădescu

Se cere: $\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$, $\forall n \geq 1$ 8

E_1) I: $P(1)$: $\frac{1}{2} < \frac{1}{\sqrt{3}} \Big| \cdot 2\sqrt{3} \Rightarrow \sqrt{3} < 2$ adev.

E_2) II Fie $P(k)$ adev. $\overset{(?)}{\Longrightarrow} P(k+1)$ adev.

$P(k)$: $\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k} < \frac{1}{\sqrt{2k+1}}$

$P(k+1)$: $\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{2k+3}}$

E_3) Știm că $P(k)$ adev. $\Rightarrow$

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k} < \frac{1}{\sqrt{2k+1}} \Big| \cdot \frac{2k+1}{2k+2}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{2k+1}}\cdot\frac{2k+1}{2k+2}$$

E_4) Suficient să arătăm că

$$\frac{1}{\sqrt{2k+1}}\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{2k+3}} \Big| ()^2$$

$$\Rightarrow \frac{1}{2k+1}\cdot\frac{(2k+1)^2}{(2k+2)^2} < \frac{1}{2k+3}$$

$$\frac{2k+1}{(2k+2)^2} < \frac{1}{2k+3}$$

$$(2k+1)(2k+3) < (2k+2)^2$$

$$4k^2+6k+2k+3 < 4k^2+8k+4$$

$3 < 4$ adev. $\Rightarrow P(k+1)$ adev.

$\Rightarrow P(n)$ adev., $\forall n \geq 1$

Prof. Ovidiu Rădescu

Arătați că $2^n \geq n^3$, $\forall n \geq 10$

E1) I $P(10): 2^{10} \geq 10^3 \Leftrightarrow 1024 \geq 1000$ adev.

E2) II Pp. $P(k)$ adev. $\overset{(?!)}{\Longrightarrow} P(k+1)$ adev.

$P(k): 2^k \geq k^3$

$P(k+1): 2^{k+1} \geq (k+1)^3$

E3) Știm că $P(k)$ adev. $\Rightarrow 2^k \geq k^3 \mid \cdot 2$

$\Rightarrow 2^{k+1} \geq 2k^3$

E4) este suficient să arăt că $2k^3 \geq (k+1)^3$

$\Leftrightarrow 2k^3 \geq k^3 + 3k^2 + 3k + 1$

$k^3 \geq 3k^2 + 3k + 1 \mid : k^2$ ≠ a folosi $k \geq 10$

E5) $k \geq 3 + 3 \cdot \frac{1}{k} + \frac{1}{k^2}$

E6) Dar, $k \geq 10 \Rightarrow \frac{1}{k} \leq \frac{1}{10}$ și $\frac{1}{k^2} \leq \frac{1}{100}$

$\Rightarrow 3 + 3 \cdot \frac{1}{k} + \frac{1}{k^2} \leq 3 + \frac{3}{10} + \frac{1}{100} < 5$

$k \geq 10$

$\Rightarrow k \geq 3 + 3 \cdot \frac{1}{k} + \frac{1}{k^2} \Rightarrow P(k+1)$ adev.

$\Rightarrow P(n)$ adev, $\forall n \geq 10$

Prof. Ovidiu Badescu

Arătați că:

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{3n+1}>1,\ \forall n\in\mathbb{N}^*$$

$E_1)$ I $P(1)$: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>1 \mid\cdot 12 \Leftrightarrow 6+4+3>12$ adev.

$E_2)$ II $P(k)$: $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

$P(k)$: $\frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{3k+1}>1$

$P(k+1)$: $\frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{3k+4}>1$

$E_3)$ Știm că $P(k)$ adev. $\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{3k+1}>1$

Trebuie să obținem $P(k+1)$

$$\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{3k+1}>1 \Big| -\frac{1}{k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$$

$$\Rightarrow \frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{3k+4}>1-\frac{1}{k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$$

$E_4)$ Este suficient să arătăm că

$$1-\frac{1}{k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}>1$$

$$\Leftrightarrow -\frac{2}{3k+3}+\frac{1}{3k+2}+\frac{1}{3k+4}>0 \Leftrightarrow \frac{1}{3k+2}+\frac{1}{3k+4}>\frac{2}{3k+3}$$

$$\Leftrightarrow \frac{6k+6}{(3k+2)(3k+4)}>\frac{2}{3k+3} \Leftrightarrow 6(k+1)\cdot 3(k+1)>2(3k+2)\cdot(3k+4)$$

$\Leftrightarrow 18k^2+36k+18>18k^2+36k+16$ adev.

$\Rightarrow P(n)$ adev., $\forall n\geq 1$

Arătați că:

$$\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n-2}>\frac{1}{4},\ \forall n\geq 3$$

$E_1)$ $P(3):\ \frac{1}{3}+\frac{1}{4}>\frac{1}{4}$ adev.

$E_2)$ $\exists n.\ P(k)$ adev. $\overset{(?)}{\Longrightarrow} P(k+1)$ adev.

$$P(k):\ \frac{1}{k}+\frac{1}{k+1}+\dots+\frac{1}{2k-2}>\frac{1}{4}$$

$$P(k+1):\ \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{1}{4}$$

$E_3)$ Știm că $P(k)$ adev. $\Rightarrow$

$$\Rightarrow \frac{1}{k}+\frac{1}{k+1}+\dots+\frac{1}{2k-2}>\frac{1}{4}\ \Big|-\frac{1}{k}+\frac{1}{2k-1}+\frac{1}{2k}$$

$$\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{1}{4}-\frac{1}{k}+\frac{1}{2k-1}+\frac{1}{2k}$$

$E_4)$ Este suficient să arăt. că

$$\frac{1}{4}-\frac{2}{k}+\frac{1}{2k-1}+\frac{1}{2k}>\frac{1}{4}$$

$$\Leftrightarrow -\frac{1}{2k}+\frac{1}{2k-1}>0 \ (\Leftrightarrow \frac{1}{2k-1}>\frac{1}{2k}$$

$\Leftrightarrow 2k>2k-1$ adev. $\Rightarrow P(k+1)$ adev. $\Rightarrow$

$\Rightarrow P(n)$ adev., $\forall n\geq 3$

Prof. Ovidiu Bădescu

Arătați că:

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{4n}>1,\ \forall n\geq 1$$

E1) I P(1): $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>1 \mid \cdot 12$

$\Rightarrow 6+4+3>12 \Leftrightarrow 13>12$ Adev.

E2) II Pp. P(k) adev. $\overset{(?)}{\Longrightarrow}$ P(k+1) adev.

P(k): $\frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{4k}>1$

P(k+1): $\frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{4k+4}>1$

E3) Știu că P(k) adev.

$$\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{4k}>1 \Big| -\frac{1}{k+1}+\frac{1}{4k+1}+\frac{1}{4k+2}+\frac{1}{4k+3}+\frac{1}{4k+4}$$

$$\Rightarrow \frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{4k+4}>1-\frac{1}{k+1}+\frac{1}{4k+1}+\frac{1}{4k+2}+\frac{1}{4k+3}+\frac{1}{4k+4}$$

E4) E suficient să arătăm că:

$$1-\underbrace{\frac{4}{k+1}+\frac{1}{4k+1}+\frac{1}{4k+2}+\frac{1}{4k+3}+\frac{1}{4k+4}}>1$$

$$\Rightarrow \frac{-3}{4k+4}+\frac{1}{4k+1}+\frac{1}{4k+2}+\frac{1}{4k+3}>0$$

$$\Rightarrow \frac{1}{4k+1}+\frac{1}{4k+2}+\frac{1}{4k+3}>\frac{3}{4k+4} \Big| \cdot (4k+1)(4k+2)(4k+3)(4k+4)$$

$$\Rightarrow (4k+2)(4k+3)(4k+4)+(4k+1)(4k+3)(4k+4)+(4k+1)(4k+2)(4k+3)>3(4k+1)(4k+2)(4k+3)$$

Se calculează (urâte) și iese relație adevărată

Prof. Ovidiu Badescu

Demonstrați că:

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}>\frac{1}{5},\ \forall n\geq 2$$

E1) I $P(2)$: $\frac{1}{3}+\frac{1}{4}>\frac{1}{5}\ |\cdot 60$

$\Rightarrow 20+15>12$ adev.

II Fie $P(k)$ adev. $\overset{(?)}{\Rightarrow} P(k+1)$ adev.

unde $P(k)$: $\frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{1}{5}$

$P(k+1)$: $\frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k+2}>\frac{1}{5}$

E2) Știm că $P(k)$ adev $\Rightarrow \frac{1}{k+1}+\frac{1}{k+2}+\dots+\frac{1}{2k}>\frac{1}{5}\ \Big|-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$

$$\Rightarrow \frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k+2}>\frac{1}{5}-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

E suficient să arătăm că:

$$\frac{1}{5}-\frac{2}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}<\frac{1}{5}$$

$$\Leftrightarrow -\frac{1}{2k+2}+\frac{1}{2k+1}>0 \Leftrightarrow \frac{1}{2k+1}>\frac{1}{2k+2}$$

$\Leftrightarrow 2k+2>2k+1$ adev

$\Rightarrow P(k+1)$ adev $\Rightarrow P(n)$ adev., $\forall n\geq 1$

Prof. Ovidiu Bădescu