CALCULAREA SUMELOR DE FRACȚII

Calculați, atât prin prelucrarea numărătorului cât și prin metoda coeficienților nedeterminați următoarele sume:

1) $S=\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\dots+\frac{1}{(3n-2)(3n+1)}$

2) $S=\frac{1}{1\cdot 5}+\frac{1}{5\cdot 7}+\dots+\frac{1}{(4n+1)(4n+5)}$

3) $S=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\dots+\frac{1}{(2n+1)(2n+3)}$

4) $S=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{(n-1)n}$

5) $S=\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\dots+\frac{1}{(3n+2)(3n+5)}$

6) $S=\frac{1}{3\cdot 7}+\frac{1}{7\cdot 11}+\dots+\frac{1}{(4n-1)(4n+3)}$

7) $S=\frac{1}{3\cdot 8}+\frac{1}{8\cdot 13}+\dots+\frac{1}{(5n-2)(5n+3)}$

8)

Calculați, prin prelucrarea numărătorului

1) $S=\sum_{k=1}^{n}\frac{k^4+k^2+1}{k^4+k}$

Prof. Ovidiu Bădescu

Calculați $S=\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots+\frac{1}{(3n-2)(3n+1)}$ 1

met I (gimnaziu, nu e generală)

E1) $S=\frac{1}{3}\left(\frac{4-1}{1\cdot 4}+\frac{7-4}{4\cdot 7}+\ldots+\frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)}\right)$

E2) $S=\frac{1}{3}\left(\frac{4}{1\cdot 4}-\frac{1}{1\cdot 4}+\frac{7}{4\cdot 7}-\frac{4}{4\cdot 7}+\ldots+\frac{3n+1}{(3n-2)(3n+1)}-\frac{3n-2}{(3n-2)(3n+1)}\right)$,

$S=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\ldots+\frac{1}{3n-2}-\frac{1}{3n+1}\right)$

$S=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)=\frac{1}{3}\cdot\frac{3n+1-1}{3n+1}=\frac{n}{3n+1}$

met II (generală, metoda coef. nedeterminați)

E1) $S=\sum_{k=1}^{n}\frac{1}{(3k-2)(3k+1)}$ E2) $\frac{1}{(3k-2)(3k+1)}=\frac{\frac{3k+1}{A}}{3k-2}+\frac{\frac{3k-2}{B}}{3k+1}=$

$=\frac{A(3k+1)+B(3k-2)}{(3k-2)(3k+1)}=\frac{3kA+A+3kB+2B}{(3k-2)(3k+1)}=\frac{k(3A+3B)+A+2B}{(3k-2)(3k+1)}$

E3) egalez coef de la numărător $\Rightarrow\begin{cases}3A+3B=0\ |:3\\ A+2B=1\end{cases}$

$\Rightarrow\begin{cases}A+B=0\\ A-2B=1\end{cases}$

„-": $3B=-1\Rightarrow\boxed{B=-1/3}$ deci, $A+B=0\Rightarrow\boxed{A=1/3}$

E4) $\frac{1}{(3k-2)(3k+1)}=\frac{\frac{1}{3}}{3k-2}+\frac{-\frac{1}{3}}{3k+1}=\frac{1}{3}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)$

E5) $\sum_{k=1}^{n}\frac{1}{(3k-2)(3k+1)}=\sum_{k=1}^{n}\frac{1}{3}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)=$

$=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\ldots+\frac{1}{3n-2}-\frac{1}{3n+1}\right)=$

$=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)=\frac{1}{3}\cdot\frac{3n+1-1}{3n+1}=\frac{1}{3}\cdot\frac{3n}{3n+1}=\frac{n}{3n+1}$

Prof. Ovidiu Bârsan

Calculați:

$$S=\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\dots+\frac{1}{(4n+1)(4n+5)}$$

met I

$$E_1)\ S=\frac{1}{4}\left(\frac{5-1}{1\cdot 5}+\frac{9-5}{5\cdot 9}+\dots+\frac{(4n+5)-(4n+1)}{(4n+1)(4n+5)}\right)$$

$$S=\frac{1}{4}\left(\frac{1}{1}-\frac{1}{5}+\frac{1}{5}-\frac{1}{9}+\dots+\frac{1}{4n+1}-\frac{1}{4n+5}\right)$$

$$S=\frac{1}{4}\left(1-\frac{1}{4n+5}\right)=\frac{1}{4}\cdot\frac{4n+5-1}{4n+5}=\frac{4n+4}{4(4n+5)}=\frac{n+1}{4n+5}$$

met II

$$E_1)\ S=\sum_{k=0}^{n}\frac{1}{(4k+1)(4k+5)}$$

$$E_2)\ \frac{1}{(4k+1)(4k+5)}=\frac{A}{4k+1}+\frac{B}{4k+5}=\frac{A(4k+5)+B(4k+1)}{(4k+1)(4k+5)}=$$

$$=\frac{k(4A+4B)+(5A+B)}{(4k+1)(4k+5)}$$ egalăm coef de la numărăt

$$E_3)\ \begin{cases}4A+4B=0\ |:4\\ 5A+B=1\end{cases}\Rightarrow\begin{cases}A+B=0\\ 5A+B=1\end{cases}$$

$$\overset{-}{4}\ \overset{-}{1}:\ -4A=-1\Rightarrow A=\frac{1}{4}\Rightarrow B=-\frac{1}{4}$$

$$E_4)\ S=\sum_{k=0}^{n}\left(\frac{\frac{1}{4}}{4k+1}+\frac{-\frac{1}{4}}{4k+5}\right)=\sum_{k=0}^{n}\frac{1}{4}\left(\frac{1}{4k+1}-\frac{1}{4k+5}\right)$$

$$E_5)\ S=\frac{1}{4}\left(1-\frac{1}{5}+\frac{1}{5}-\frac{1}{9}+\dots+\frac{1}{4n+1}-\frac{1}{4n+5}\right)$$

$$S=\frac{1}{4}\left(1-\frac{1}{4n+5}\right)=\frac{1}{4}\cdot\frac{4n+5-1}{4n+5}=\frac{4(n+1)}{4(4n+5)}$$

$$S=\frac{n+1}{4n+5}$$

Prof. Ovidiu Bădescu

Calculați $S = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \dots + \frac{1}{(2n+1)(2n+3)}$ prin prelucrarea numărătorului și metoda coef nedeterm.

met I:

E1) $S = \sum_{k=0}^{n} \frac{1}{(2k+1)(2k+3)}$

E2) $\frac{1}{(2k+1)(2k+3)} = \frac{A}{2k+1} + \frac{B}{2k+3} = \frac{k\cdot(2A+2B)+3A+B}{(2k+1)(2k+3)}$

$\Rightarrow \begin{cases} 2A+2B=0 \mid :2 \\ 3A+B=1 \end{cases} \Rightarrow \begin{cases} A+B=0 \mid \cdot(-1) \\ 3A+B=1 \end{cases}$

„+": $2A=1 \Rightarrow A=\frac{1}{2}$

Dar, $A+B=0 \Rightarrow B=-\frac{1}{2}$

E3) $\frac{1}{(2k+1)(2k+3)} = \frac{\frac{1}{2}}{2k+1} + \frac{-\frac{1}{2}}{2k+3} = \frac{1}{2}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$

$k=0 \Rightarrow \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$

$k=1 \Rightarrow \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

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$k=n \Rightarrow \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$

„+": $S = \frac{1}{2}\left(1 - \frac{1}{2n+3}\right) = \frac{1}{2}\cdot\frac{2n+3-1}{2n+3} = \frac{1}{2}\cdot\frac{2n+2}{2n+3} =$

$S = \frac{1}{2}\cdot\frac{2(n+1)}{2n+3} = \frac{n+1}{2n+3}$

met II: $S = \frac{1}{2}\left(\frac{3-1}{1\cdot 3} + \frac{5-3}{3\cdot 5} + \dots + \frac{(2n+3)-(2n+1)}{(2n+1)(2n+5)}\right)$

$S = \frac{1}{2}\left(\frac{3}{1\cdot 3} - \frac{1}{1\cdot 3} + \frac{5}{3\cdot 5} - \frac{3}{3\cdot 5} + \dots + \frac{2n+3}{(2n+1)(2n+3)} - \frac{2n+1}{(2n+1)(2n+3)}\right)$

$S = \frac{1}{2}\left(1 - \frac{1}{2n+3}\right) = \frac{1}{2}\cdot\frac{2n+3-1}{2n+3} = \frac{n+1}{2n+3}$

Prof. Dragu Radescu

Calculați $S = \frac{1}{3\cdot 7} + \frac{1}{7\cdot 11} + \dots + \frac{1}{(4n-1)(4n+3)}$

met I: (gimnazii)

$E_1)\ S = \frac{1}{4}\left(\frac{7-3}{3\cdot 7} + \frac{11-7}{7\cdot 11} + \dots + \frac{(4n+3)-(4n-1)}{(4n-1)(4n+3)}\right)$

$S = \frac{1}{4}\left(\frac{7}{3\cdot 7} - \frac{3}{3\cdot 7} + \frac{11}{7\cdot 11} - \frac{7}{7\cdot 11} + \dots + \frac{4n+3}{(4n-1)(4n+3)} - \frac{4n-1}{(4n-1)(4n+3)}\right)$

$S = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{7} + \frac{1}{7} - \frac{1}{11} + \dots + \frac{1}{4n-1} - \frac{1}{4n+3}\right)$

$S = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{4n+3}\right) = \frac{1}{4}\cdot\frac{4n+3-3}{3(4n+3)} = \frac{n}{3(4n+3)}$

met II (liceu)

$E_1)\ S = \sum_{k=1}^{n} \frac{1}{(4k-1)(4k+3)}$

$E_2)\ \frac{1}{(4k-1)(4k+3)} = \frac{4k+3}{4k-1}\cdot\frac{A}{} + \frac{4k-1}{4k+3}\cdot\frac{B}{} = \frac{4Ak+3A+4Bk-B}{(4k-1)(4k+3)} =$

$= \frac{k(4A+4B)+3A-B}{(4k-1)(4k+3)}$

$E_3)$ egalăm coeficienții $\Rightarrow \begin{cases} 4A+4B=0 \mid :4 \\ 3A-B=1 \end{cases}$

$\Rightarrow \begin{cases} A+B=0 \\ 3A-B=1 \end{cases}$

„+": $4A=1 \Rightarrow A=\frac{1}{4}$ înlocuim $A+B=0 \Rightarrow B=-\frac{1}{4}$

$E_4)\ \frac{1}{(4k-1)(4k+3)} = \frac{\frac{1}{4}}{4k-1} - \frac{\frac{1}{4}}{4k+3} = \frac{1}{4}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$

$E_5)\ k=1 \Rightarrow \frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\right)$

$k=2 \Rightarrow \frac{1}{4}\left(\frac{1}{7} - \frac{1}{11}\right)$

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$k=n \Rightarrow \frac{1}{4}\left(\frac{1}{4n-1} - \frac{1}{4n+3}\right)$

$S = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{4n+3}\right) = \frac{1}{4}\cdot\frac{4n+3-3}{3(4n+3)} = \frac{n}{3(4n+3)}$

Calculați $S=\frac{1}{3\cdot 8}+\frac{1}{8\cdot 13}+\dots+\frac{1}{(5n-2)(5n+3)}$ 7

met I: prelucrarea numărătorului (nu e metodă generală)

E1) $S=\frac{1}{5}\left(\frac{8-3}{3\cdot 8}+\frac{13-8}{8\cdot 13}+\dots+\frac{(5n+3)-(5n-2)}{(5n-2)(5n+3)}\right)$

$S=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{8}+\frac{1}{8}-\frac{1}{13}+\dots+\frac{1}{5n-2}-\frac{1}{5n+3}\right)$

$S=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{5n+3}\right)=\frac{1}{5}\cdot\frac{5n+3-3}{3(5n+3)}=\frac{5n}{5\cdot 3(5n+3)}$

$S=\frac{n}{3(5n+3)}$

met II (metoda coef. nedeterminați)

E1) $S=\sum_{k=1}^{n}\frac{1}{(5k-2)(5k+3)}$

E2) $\frac{1}{(5k-2)(5k+3)}=\frac{A}{5k-2}+\frac{B}{5k+3}=\frac{A(5k+3)+B(5k-2)}{(5k-2)(5k+3)}=$

$=\frac{5kA+3A+5kB-2B}{(5k-2)(5k+3)}=\frac{k(5A+5B)+3A-2B}{(5k-2)(5k+3)}$

E3) egalez coef de la numărător $\Rightarrow\begin{cases}5A+5B=0 & |:5\\ 3A-2B=1\end{cases}$

$\Rightarrow\begin{cases}A+B=0 & |\cdot 2\\ 3A-2B=1\end{cases}\Rightarrow$

„+" $5A=1\Rightarrow\boxed{A=\frac{1}{5}}\Rightarrow A+B=0\Rightarrow\boxed{B=-\frac{1}{5}}$

E4) $S=\sum_{k=1}^{n}\frac{1}{(5k-2)(5k+3)}=\sum_{k=1}^{n}\left(\frac{\frac{1}{5}}{5k-2}+\frac{-\frac{1}{5}}{5k+3}\right)=$

$=\sum_{k=1}^{n}\frac{1}{5}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{8}+\frac{1}{8}-\frac{1}{13}+\dots+\frac{1}{5n-2}-\frac{1}{5n+3}\right)$

$\Rightarrow S=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{5n+3}\right)=\frac{1}{5}\cdot\frac{5n+3-3}{3(5n+3)}=\frac{n}{3(5n+3)}$

Prof. Ovidiu Bădescu

Calculati $S=\sum_{k=1}^{n} \frac{k^4+k^2+1}{k^4+k}$

$E_1)$ $\frac{k^4+k^2+1}{k^4+k} = \frac{k^4+2k^2+1-k^2}{k^4+k} = \frac{(k^2+1)^2-k^2}{k(k^3+1)} =$

$= \frac{(k^2+1-k)(k^2+1+k)}{k(k+1)(k^2-k+1)} = \frac{(k^2-k+1)(k^2+k+1)}{(k^2-k+1)(k^2+k)} =$

$= \frac{k^2+k+1}{k(k+1)} = \frac{k^2+2k+1-k}{k(k+1)} = \frac{(k+1)^2-k}{k(k+1)} =$

$= \frac{(k+1)^2}{k(k+1)} - \frac{k}{k(k+1)} = \frac{k+1}{k} - \frac{1}{k+1} = 1+\frac{1}{k}-\frac{1}{k+1}$

$E_2)$ $k=1 \Rightarrow 1+1-\frac{1}{2}$

$k=2 \Rightarrow 1+\frac{1}{2}-\frac{1}{3}$

$k=3 \Rightarrow 1+\frac{1}{3}-\frac{1}{4}$

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$k=n \Rightarrow 1+\frac{1}{n}-\frac{1}{n+1}$

"+" $\Rightarrow S = n+1 + 1 - \frac{1}{n+1}$

$S = \frac{n^2+n+n+1+1}{n+1} \Rightarrow$

$S = \frac{n^2+2n}{n+1}$

Prof. Ovidiu Bădescu